A Robust Filter Design for Uncertain Singular Systems with Unreliable Channels

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ABSTRACT:
This paper considers the problem of robust H_∞ filter design in uncertain discrete-time singular systems with possible missing measurements due to unreliable network transmission channels. The stochastic variable satisfying Bernoulli random binary distribution is introduced to model the missing phenomena and the corresponding filtering error dynamics with delay is then induced. We provide a set of sufficient conditions for the existence of the desired filter, and propose a robust filter design method under a strict linear matrix inequality framework. A numerical example is given to illustrate the effectiveness of the proposed method.

KEYWORDS: linear systems, filter design, linear matrix inequality, unreliable channels.

1. INTRODUCTION

The H_∞ optimal filtering problem for singular systems has been an important research topic in the past decade. This is due, not only to the theoretical interests, but also to the relevance of the topic in various engineering applications. For instance, based on the admissibility assumption of uncertain singular systems, some suboptimal H_∞ singular filter design methods were proposed in [5, 6, 7], a linear matrix inequality (LMI) based filter design approach was proposed for impulsive stochastic systems in [15], and the reduced-order H_∞ filtering problem based on the projection lemma was investigated in [9, 17], and so on. Most literature concerning filtering techniques such as in [1, 3, 5, 6, 12, 15, 17] and so forth assumed that the measurements contain consecutive useful signals. However, the measurements are not consecutive but contain missing observations in practical applications. The missing observations are caused for a variety of reasons. Take the networked operation systems, for example. The current networks induce possible data transmission loss and delay, which are two main problems in networked operation systems, due to limited bandwidth, intermittent remote sensor failures, or some of the data may be jammed or coming from a very noisy environment.

Recently, more and more efforts have been focused on the problem of H_∞ filtering for various time-delay systems, and many approaches have been proposed, including the Riccati equation approach [11], the polynomial equation approach [19], the LMI approach [1, 3, 12] and so on. In most existing works dealing with the filtering problem for time-delay systems, the measurement missing phenomena have seldom been taken into account, except [9, 13, 14]. In [9], a reduced order filter design method is considered for nominal systems. The practical applications are limited because of its corresponding non-strict LMI constraints and nominal systems formation. For discrete-time singular systems in the simultaneous presence of time delays, missing measurements, and parameter uncertainties, the problem of robust H_∞ filtering has not been fully investigated and remains to be challenging.

In this paper, the H_∞ filtering problem for a class of uncertain discrete-time singular systems with possible missing observation due to unreliable networked transmission will be considered. The purpose here is to design a stable filter such that the corresponding filtering error dynamics is the exponentially mean square admissible, and satisfies a prescribed level of H_∞ filtering performance via a set of conditions under the strictly LMI framework.

Here, we introduce some notations to be used subsequently. The inequality $\boldsymbol{P} > \boldsymbol{0}$ means that the matrix $\boldsymbol{P}$ is symmetric and positive definite, and $\boldsymbol{P} > \boldsymbol{Q}$ means $\boldsymbol{P} - \boldsymbol{Q} > \boldsymbol{0}$. Similar definitions apply to symmetric positive/negative semi-definite matrices. $\boldsymbol{I}_m$ is the identity matrix with dimension m. Matrices are assumed to have compatible dimensions for algebraic operations if their dimensions are not explicitly stated. $diag(\boldsymbol{M}_1, \boldsymbol{M}_2)$ is the block diagonal matrix with diagonal elements $\boldsymbol{M}_1$, $\boldsymbol{M}_2$. The superscript $^{\mathrm{T}}$ represents the transpose of a matrix. $l_2[0,\infty)$ is the space of square-summable vectors. Prob{·} represents the probability of the occurrence of an event, and E{·} denotes the expectation operator with respect to some probability measure. Finally, $*$ is used to simplify the presentation of symmetric matrices.

The remainder of this paper is organized as follows. In Section 2 we give some preliminaries about singular systems and formulate the filtering problem with unreliable networked channels. Section 3 provides a sufficient condition for the existence of the corresponding filter, and develops an LMI-based method to the problem. Section 4 gives a numerical example to show validity of the proposed method, and finally, Section 5 concludes the paper.

2. PROBLEM STATEMENT AND DEFINITIONS

Consider the following nominal singular system,

$$\Sigma_0\colon \begin{cases} \mathrm{E}_0\bar{\mathrm{x}}(\mathrm{k}+1) = \mathrm{A}_0\bar{\mathrm{x}}(\mathrm{k}) + \mathrm{B}_0\mathrm{u}(\mathrm{k}) \\ \bar{\mathrm{z}}(\mathrm{k}) = \mathrm{L}_0\bar{\mathrm{x}}(\mathrm{k}), \end{cases} \quad (1)$$

where $\bar{\boldsymbol{x}}(k) \in \mathcal{R}^n$ and $rank\boldsymbol{E}_0 = r < n$. The unforced singular system pair $(\boldsymbol{E}_0, \boldsymbol{A}_0)$ of (1) with $\boldsymbol{u}(k) \equiv \boldsymbol{0}$ is *regular*, if $det(z\boldsymbol{E}_0 - \boldsymbol{A}_0)$ is not identically zero. If $deg(det(z\boldsymbol{E}_0 - \boldsymbol{A}_0)) = rank\boldsymbol{E}_0$, then $(\boldsymbol{E}_0, \boldsymbol{A}_0)$ is said to be *causal*. The pair $(\boldsymbol{E}_0, \boldsymbol{A}_0)$ is stable if all the roots of $det(z\boldsymbol{E}_0 - \boldsymbol{A}_0) = 0$ have magnitudes less than unity. Finally, $(\boldsymbol{E}_0, \boldsymbol{A}_0)$ is *admissible* if it is regular, causal, and stable [2].

Definition 1. [18] The singular system (1) is said to be exponentially mean-square stable if with $\boldsymbol{u}(k) = \boldsymbol{0}$, there exist constants $\alpha > 0$ and $\rho \in (0,1)$, such that

$$\mathrm{E}\{\| \bar{\boldsymbol{x}}(k) \|^2\} \le \alpha\rho^k \mathrm{E}\{\| \bar{\boldsymbol{x}}(0) \|^2\}$$

for all $k \in \boldsymbol{Z}^+$, where $\boldsymbol{Z}^+$ denotes the set of positive integers.

Definition 2. The singular system (1) is exponentially mean-square admissible if it is regular, casual, and exponentially mean-square stable.

Consider the following uncertain networked filtering system with measurements communicated from unreliable networks or remote sensors showing in Fig. 1.

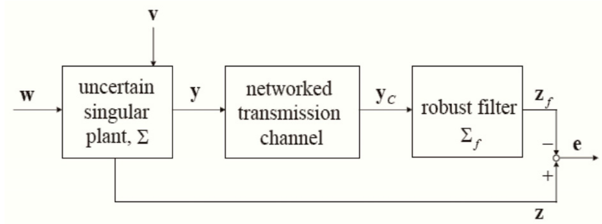

Fig. 1. Networked filtering systems with unreliable channels

The singular system is determined as in (2).

$$\Sigma\colon \begin{cases} \boldsymbol{E}\boldsymbol{x}(k+1) = \boldsymbol{A}_\delta\boldsymbol{x}(k) + \boldsymbol{B}_\delta\boldsymbol{w}(k) \\ \boldsymbol{y}(k) = \boldsymbol{C}_\delta\boldsymbol{x}(k) + \boldsymbol{D}_\delta\boldsymbol{v}(k) \\ \boldsymbol{z}(k) = \boldsymbol{L}_\delta\boldsymbol{x}(k) + \boldsymbol{J}_\delta\boldsymbol{w}(k), \end{cases} \quad (2)$$

where

$$\begin{aligned} &\boldsymbol{A}_\delta = \boldsymbol{A} + \delta\boldsymbol{A}, \quad \boldsymbol{B}_\delta = \boldsymbol{B} + \delta\boldsymbol{B}, \\ &\boldsymbol{C}_\delta = \boldsymbol{C} + \delta\boldsymbol{C}, \quad \boldsymbol{D}_\delta = \boldsymbol{D} + \delta\boldsymbol{D}, \\ &\boldsymbol{L}_\delta = \boldsymbol{L} + \delta\boldsymbol{L}, \quad \boldsymbol{J}_\delta = \boldsymbol{J} + \delta\boldsymbol{J}, \end{aligned} \quad (3)$$

and $\boldsymbol{x}(k) \in \mathcal{R}^n$ is the state vector, $\boldsymbol{y}(k) \in \mathcal{R}^p$ is the measured output vector which is transmitted to a filter via unreliable networks, $\boldsymbol{z}(k) \in \mathcal{R}^q$ is the vector to be estimated, and $\boldsymbol{w}(k), \boldsymbol{v}(k) \in \mathcal{R}^m$ are disturbance input vector and measured noise, respectively, in $l_2[0,\infty)$, which is the space of square-summable vectors. The matrix $\boldsymbol{E} \in \mathcal{R}^{n\times n}$ is singular with $rank\boldsymbol{E} = r < n$, and $\boldsymbol{A}$, $\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{L}$, $\boldsymbol{J}$ are known real constant matrices with appropriate dimensions. The constant uncertainty matrices satisfy

$$\begin{bmatrix} \delta\boldsymbol{A} & \delta\boldsymbol{B} \\ \delta\boldsymbol{C} & \delta\boldsymbol{D} \\ \delta\boldsymbol{L} & \delta\boldsymbol{J} \end{bmatrix} = \begin{bmatrix} \boldsymbol{H}_x \\ \boldsymbol{H}_y \\ \boldsymbol{H}_z \end{bmatrix} \Delta[\,\boldsymbol{F}_x \quad \boldsymbol{F}_u\,] \quad (4)$$

with $\Delta^{\mathrm{T}}\Delta \le \boldsymbol{I}$ and $\Delta \in \mathcal{R}^{d_1\times d_2}$. Assume that the pair $(\boldsymbol{E}, \boldsymbol{A} + \delta\boldsymbol{A})$ is admissible. The measurements, which may contain missing data due to the transmission via unreliable networks, are described by

$$\boldsymbol{y}_c(k) = (1-\alpha_k)\boldsymbol{y}(k) + \alpha_k\boldsymbol{y}(k-1), \quad (5)$$

where the stochastic variable $\alpha_k \in \mathcal{R}$ is a Bernoulli distributed white sequence taking the values of 1 and 0 with

$$\mathrm{Prob}\{\alpha_k = 1\} \equiv \mathrm{E}\{\alpha_k\} = \eta, \quad (6)$$

$$\mathrm{Prob}\{\alpha_k = 0\} = 1 - \mathrm{E}\{\alpha_k\} = 1 - \eta, \quad (7)$$

$\eta \in [0,1]$ is a known constant, $\alpha_k = 1$ represents the data-loss event at k, while $\alpha_k = 0$ means that data are received at k. Prob{·} is the probability of the occurrence of an event, and E{·} denotes the expectation operator with respect to some probability measure. Model (5) shows that the data dropouts can be encountered in data transmission in networked control systems. It means that at least one of these measurements is received by the filter, which is usually required in practice.

To estimated $\boldsymbol{z}(k)$, the following filter

$$\Sigma_f: \begin{cases} \boldsymbol{x}_f(k+1) = \boldsymbol{A}_f \boldsymbol{x}_f(k) + \boldsymbol{B}_f \boldsymbol{y}_c(k) \\ \boldsymbol{z}_f(k) = \boldsymbol{C}_f \boldsymbol{x}_f(k) + \boldsymbol{D}_f \boldsymbol{y}_c(k), \end{cases} \tag{8}$$

is adopted, where $\boldsymbol{x}_f(k) \in \mathcal{R}^n$ and $\boldsymbol{z}_f(k) \in \mathcal{R}^q$. The matrices $\boldsymbol{A}_f, \boldsymbol{B}_f, \boldsymbol{C}_f$, and $\boldsymbol{D}_f$ are to be determined. Assume $\{\boldsymbol{x}_e(-1), \boldsymbol{w}_e(-1)\} = \boldsymbol{0}$. From Σ in (2) and Σ_f in (8), the filtering error dynamics may be written as

$$\Sigma_e: \begin{cases} \boldsymbol{E}_e \boldsymbol{x}_e(k+1) = \widetilde{\boldsymbol{A}}_e \boldsymbol{x}_e(k) + \widetilde{\boldsymbol{B}}_e \boldsymbol{w}_e(k) + \\ \quad \widetilde{\boldsymbol{A}}_{ed} \boldsymbol{x}_e(k-1) + \widetilde{\boldsymbol{B}}_{ed} \boldsymbol{w}_e(k-1), \\ \\ \boldsymbol{e}(k) = \widetilde{\boldsymbol{C}}_e \boldsymbol{x}_e(k) + \widetilde{\boldsymbol{D}}_e \boldsymbol{w}_e(k) + \\ \quad \widetilde{\boldsymbol{C}}_{ed} \boldsymbol{x}_e(k-1) + \widetilde{\boldsymbol{D}}_{ed} \boldsymbol{w}_e(k-1), \end{cases} \tag{9}$$

where $\boldsymbol{e}(k) = \boldsymbol{z}(k) - \boldsymbol{z}_f(k)$, $\boldsymbol{x}_e^{\mathrm{T}}(k) = [\,\boldsymbol{x}^{\mathrm{T}}(k) \;\; \boldsymbol{x}_f^{\mathrm{T}}(k)\,]$, $\boldsymbol{w}_e^{\mathrm{T}}(k) = [\,\boldsymbol{w}^{\mathrm{T}}(k) \;\; \boldsymbol{v}^{\mathrm{T}}(k)\,]$, and $\boldsymbol{E}_e = diag(\boldsymbol{E}, \;\boldsymbol{I}_n)$,

$$\begin{aligned} \widetilde{\boldsymbol{A}}_e &= \boldsymbol{A}_e + \delta\boldsymbol{A}_e, & \widetilde{\boldsymbol{B}}_e &= \boldsymbol{B}_e + \delta\boldsymbol{B}_e, \\ \widetilde{\boldsymbol{A}}_{ed} &= \boldsymbol{A}_{ed} + \delta\boldsymbol{A}_{ed}, & \widetilde{\boldsymbol{B}}_{ed} &= \boldsymbol{B}_{ed} + \delta\boldsymbol{B}_{ed}, \\ \widetilde{\boldsymbol{C}}_e &= \boldsymbol{C}_e + \delta\boldsymbol{C}_e, & \widetilde{\boldsymbol{D}}_e &= \boldsymbol{D}_e + \delta\boldsymbol{D}_e, \\ \widetilde{\boldsymbol{C}}_{ed} &= \boldsymbol{C}_{ed} + \delta\boldsymbol{C}_{ed}, & \widetilde{\boldsymbol{D}}_{ed} &= \boldsymbol{D}_{ed} + \delta\boldsymbol{D}_{ed}. \end{aligned} \tag{10}$$

Let $\bar{\alpha}_k = 1 - \alpha_k$, $\boldsymbol{H} = [\boldsymbol{I}_n \;\; \boldsymbol{0}] \in \mathcal{R}^{n\times 2n}$, and $\boldsymbol{H}_b = [\boldsymbol{0} \;\; \boldsymbol{I}_m] \in \mathcal{R}^{m\times 2m}$ for brevity. We get

$$\begin{aligned} \boldsymbol{A}_e + \delta\boldsymbol{A}_e &= \begin{bmatrix} \boldsymbol{A} + \delta\boldsymbol{A} & \boldsymbol{0} \\ \bar{\alpha}_k \boldsymbol{B}_f(\boldsymbol{C} + \delta\boldsymbol{C}) & \boldsymbol{A}_f \end{bmatrix}, \\ \boldsymbol{B}_e + \delta\boldsymbol{B}_e &= \begin{bmatrix} \boldsymbol{B} + \delta\boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{0} & \bar{\alpha}_k \boldsymbol{B}_f(\boldsymbol{D} + \delta\boldsymbol{D}) \end{bmatrix}, \\ \boldsymbol{C}_e + \delta\boldsymbol{C}_e &= [\,\boldsymbol{L} + \delta\boldsymbol{L} - \bar{\alpha}_k \boldsymbol{D}_f(\boldsymbol{C} + \delta\boldsymbol{C}) \quad -\boldsymbol{C}_f], \\ \boldsymbol{D}_e + \delta\boldsymbol{D}_e &= [\boldsymbol{J} + \delta\boldsymbol{J} \quad -\bar{\alpha}_k \boldsymbol{D}_f(\boldsymbol{D} + \delta\boldsymbol{D})], \end{aligned} \tag{11}$$

and present other matrices in (10) more explicitly as in (12).

$$\begin{aligned} \boldsymbol{A}_{ed} + \delta\boldsymbol{A}_{ed} &= \bar{\boldsymbol{A}}_{ed}^{\delta} \boldsymbol{H}, \\ \boldsymbol{B}_{ed} + \delta\boldsymbol{B}_{ed} &= \bar{\boldsymbol{B}}_{ed}^{\delta} \boldsymbol{H}_b, \\ \boldsymbol{C}_{ed} + \delta\boldsymbol{C}_{ed} &= \bar{\boldsymbol{C}}_{ed}^{\delta} \boldsymbol{H}, \\ \boldsymbol{D}_{ed} + \delta\boldsymbol{D}_{ed} &= \bar{\boldsymbol{D}}_{ed}^{\delta} \boldsymbol{H}_b, \end{aligned} \tag{12}$$

where

$$\begin{aligned} \bar{\boldsymbol{A}}_{ed}^{\delta} &= \overline{\boldsymbol{A}}_{ed} + \delta\overline{\boldsymbol{A}}_{ed} = \begin{bmatrix} \boldsymbol{0} \\ \alpha_k \boldsymbol{B}_f(\boldsymbol{C} + \delta\boldsymbol{C}) \end{bmatrix}, \\ \bar{\boldsymbol{B}}_{ed}^{\delta} &= \overline{\boldsymbol{B}}_{ed} + \delta\overline{\boldsymbol{B}}_{ed} = \begin{bmatrix} \boldsymbol{0} \\ \alpha_k \boldsymbol{B}_f(\boldsymbol{D} + \delta\boldsymbol{D}) \end{bmatrix}, \\ \bar{\boldsymbol{C}}_{ed}^{\delta} &= \overline{\boldsymbol{C}}_{ed} + \delta\overline{\boldsymbol{C}}_{ed} = -\alpha_k \boldsymbol{D}_f(\boldsymbol{C} + \delta\boldsymbol{C}), \\ \bar{\boldsymbol{D}}_{ed}^{\delta} &= \overline{\boldsymbol{D}}_{ed} + \delta\overline{\boldsymbol{D}}_{ed} = -\alpha_k \boldsymbol{D}_f(\boldsymbol{D} + \delta\boldsymbol{D}). \end{aligned} \tag{13}$$

Note that matrices on the left side of equations (10)-(13) are related to α_k.

The purpose here is to design a stable filter (8) such that the delay filtering error dynamics (9) is exponentially mean-square admissible with H_∞ filtering performance. It means that the filtering error singular system Σ_e will be regular, casual, and exponentially mean-square stable for all considered uncertainties, and under the zero initial condition, the filtering error will satisfy

$$\sum_{k=0}^{\infty} \mathrm{E}\{\| \boldsymbol{e}(k) \|^2\} \le \mu_e^2 \sum_{k=0}^{\infty} \| \bar{\boldsymbol{w}}(k) \|^2, \tag{14}$$

for a given scalar $\mu_e > 0$ and all nonzero $\bar{\boldsymbol{w}}_e(k)$, where $\bar{\boldsymbol{w}}_e^{\mathrm{T}}(k) = [\boldsymbol{w}_e^{\mathrm{T}}(k) \;\; \boldsymbol{v}^{\mathrm{T}}(k-1)]$.

The following lemma is useful for formulating the problem within the LMI framework.

Lemma 1. *[10] Let $\boldsymbol{\Omega}$, $\bar{\boldsymbol{M}}$, and $\bar{\boldsymbol{N}}$ be real matrices with appropriate dimensions. Then for the matrix $\bar{\boldsymbol{\Pi}}$ satisfying $\bar{\Pi}^T \bar{\Pi} \le \boldsymbol{I}$, the matrix inequality*

$$\Omega + \bar{\mathbf{M}}\bar{\Pi}\bar{\mathbf{N}} + \bar{\mathbf{N}}^{\mathrm{T}}\bar{\Pi}^{\mathrm{T}}\bar{\mathbf{M}}^{\mathrm{T}} < \mathbf{0}$$

holds if and only if there exists a scalar $\varepsilon > 0$ such that

$$\begin{bmatrix} \boldsymbol{\Omega} & \bar{\boldsymbol{M}} \\ \bar{\boldsymbol{M}}^{\mathrm{T}} & \boldsymbol{0} \end{bmatrix} + \varepsilon \begin{bmatrix} \bar{\boldsymbol{N}}^{\mathrm{T}}\bar{\boldsymbol{N}} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{I} \end{bmatrix} < \boldsymbol{0}.$$

3. ROBUST FILTER DESIGN

The following preliminary theorem, which plays a key role and is the first step toward developing an LMI solution to the problem stated above, provides a sufficient condition of exponentially mean-square admissibility and H_∞ performance for the filtering error dynamics (9).

Theorem 1. *For a given $\mu_e > 0$, the error dynamic system Σ_e in (9) is exponentially mean-square admissible and satisfies (14) for all admissible uncertainties, if there exist matrices $\boldsymbol{P}_e > \boldsymbol{0}$, $\boldsymbol{Q} > \boldsymbol{0}$, and $\boldsymbol{S}$, such that*

$$\begin{bmatrix} \Xi_{11} & * & * & * & * & * \\ \mathbf{0} & -\boldsymbol{Q} & * & * & * & * \\ \Xi_{31} & \mathbf{0} & -\mu_e^2\boldsymbol{I}_{2m} & * & * & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mu_e^2\boldsymbol{I} & * & * \\ \Xi_{51} & \Xi_{52} & \Xi_{53} & \Xi_{54} & -\boldsymbol{P}_e & * \\ \Xi_{61} & \Xi_{62} & \Xi_{63} & \Xi_{64} & \mathbf{0} & -\boldsymbol{I} \end{bmatrix} < \mathbf{0}, \quad (15)$$

where

$$\Xi_{11} = \boldsymbol{H}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{H} - \boldsymbol{E}_e^{\mathrm{T}}\boldsymbol{P}_e\boldsymbol{E}_e + \bar{\boldsymbol{A}}^{\mathrm{T}}\bar{\boldsymbol{R}} + \bar{\boldsymbol{R}}^{\mathrm{T}}\bar{\boldsymbol{A}},$$

$$\Xi_{31} = \bar{\boldsymbol{B}}^{\mathrm{T}}\bar{\boldsymbol{R}},$$

$$\Xi_{51} = \boldsymbol{P}_e(\boldsymbol{A}_e(\eta) + \delta\boldsymbol{A}_e(\eta)),$$

$$\Xi_{52} = \boldsymbol{P}_e(\overline{\boldsymbol{A}}_{ed}(\eta) + \delta\overline{\boldsymbol{A}}_{ed}(\eta)),$$

$$\Xi_{53} = \boldsymbol{P}_e(\boldsymbol{B}_e(\eta) + \delta\boldsymbol{B}_e(\eta)),$$

$$\Xi_{54} = \boldsymbol{P}_e(\overline{\boldsymbol{B}}_{ed}(\eta) + \delta\overline{\boldsymbol{B}}_{ed}(\eta)),$$

$$\Xi_{61} = \boldsymbol{C}_e(\eta) + \delta\boldsymbol{C}_e(\eta), \quad (16)$$

$$\Xi_{62} = \overline{\boldsymbol{C}}_{ed}(\eta) + \delta\overline{\boldsymbol{C}}_{ed}(\eta),$$

$$\Xi_{63} = \boldsymbol{D}_e(\eta) + \delta\boldsymbol{D}_e(\eta),$$

$$\Xi_{64} = \overline{\boldsymbol{D}}_{ed}(\eta) + \delta\overline{\boldsymbol{D}}_{ed}(\eta),$$

$$\bar{\boldsymbol{A}} = [\boldsymbol{A} + \delta\boldsymbol{A} \quad \mathbf{0}],$$

$$\bar{\boldsymbol{B}} = [\boldsymbol{B} + \delta\boldsymbol{B} \quad \mathbf{0}],$$

$$\bar{\boldsymbol{R}} = [\boldsymbol{R}\boldsymbol{S} \quad \mathbf{0}],$$

and $\boldsymbol{R} \in \mathcal{R}^{n\times(n-r)}$ *is any matrix with full column rank and satisfies* $\boldsymbol{E}^T\boldsymbol{R} = \mathbf{0}$. *The* η*-dependent matrices in (16) are defined as in (11)-(13) with* α_k *replaced by* η.

Proof: The proof of the theorem is provided in appendix.

Note that as a knack, the last two terms of $\boldsymbol{\Xi_{11}}$ in (16) improve the applicability instead of conservative of the sufficient condition, especially when it works under the LMI framework.

Theorem 2. *The filtering error dynamics* Σ_e *in (9) is exponentially mean-square admissible and satisfies (14) with all considered uncertainties, if there exist matrices* $\boldsymbol{S} \in \mathcal{R}^{(n-r)\times n}$, $\boldsymbol{W}_a \in \mathcal{R}^{n\times n}$, $\boldsymbol{W}_b \in \mathcal{R}^{n\times p}$, $\boldsymbol{W}_c \in \mathcal{R}^{q\times n}$, $\boldsymbol{D}_f \in \mathcal{R}^{q\times p}$, *and positive definite matrices* $\{\boldsymbol{X}, \Phi, \boldsymbol{Q}\} \in \mathcal{R}^{n\times n}$, *such that the inequality*

$$\begin{bmatrix} \boldsymbol{X}_{11} & * & * & * & * & * \\ \mathbf{0} & -\boldsymbol{Q} & * & * & * & * \\ \boldsymbol{X}_{31} & \mathbf{0} & -\mu_e^2\boldsymbol{I}_{2m} & * & * & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mu_e^2\boldsymbol{I} & * & * \\ \boldsymbol{X}_{51} & \boldsymbol{X}_{52} & \boldsymbol{X}_{53} & \boldsymbol{X}_{54} & \boldsymbol{X}_{55} & * \\ \boldsymbol{X}_{61} & \boldsymbol{X}_{62} & \boldsymbol{X}_{63} & \boldsymbol{X}_{64} & \mathbf{0} & -\boldsymbol{I} \end{bmatrix} < \mathbf{0} \quad (17)$$

is satisfied, where

$$\boldsymbol{X}_{11} = \begin{bmatrix} \begin{pmatrix} \boldsymbol{Q} - \boldsymbol{E}^{\mathrm{T}}\boldsymbol{X}\boldsymbol{E} + \\ \boldsymbol{A}_\delta^{\mathrm{T}}\boldsymbol{R}\boldsymbol{S} + \boldsymbol{S}^{\mathrm{T}}\boldsymbol{R}^{\mathrm{T}}\boldsymbol{A}_\delta \end{pmatrix} & * \\ (\Phi - \boldsymbol{X})\boldsymbol{E} & \Phi - \boldsymbol{X} \end{bmatrix},$$

$$\boldsymbol{X}_{31} = \begin{bmatrix} \boldsymbol{B}_\delta^{\mathrm{T}}\boldsymbol{R}\boldsymbol{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$
$$\boldsymbol{X}_{51} = \begin{bmatrix} \Phi\boldsymbol{A}_\delta & \mathbf{0} \\ \boldsymbol{X}\boldsymbol{A}_\delta + \bar{\eta}\boldsymbol{W}_b\boldsymbol{C}_\delta & \boldsymbol{W}_a \end{bmatrix},$$
$$\boldsymbol{X}_{52} = \begin{bmatrix} \mathbf{0} \\ \eta\boldsymbol{W}_b\boldsymbol{C}_\delta \end{bmatrix}, \quad (18)$$
$$\boldsymbol{X}_{53} = \begin{bmatrix} \Phi\boldsymbol{B}_\delta & \mathbf{0} \\ \boldsymbol{X}\boldsymbol{B}_\delta + \bar{\eta}\boldsymbol{W}_b\boldsymbol{D}_\delta & \bar{\eta}\boldsymbol{W}_b\boldsymbol{D}_\delta \end{bmatrix},$$
$$\boldsymbol{X}_{54} = \begin{bmatrix} \mathbf{0} \\ \eta\boldsymbol{W}_b\boldsymbol{D}_\delta \end{bmatrix}, \quad \boldsymbol{X}_{55} = -\begin{bmatrix} \Phi & \Phi \\ \Phi & \boldsymbol{X} \end{bmatrix},$$
$$\boldsymbol{X}_{61} = [\boldsymbol{L}_\delta - \bar{\eta}\boldsymbol{D}_f\boldsymbol{C}_\delta \quad -\boldsymbol{W}_c],$$
$$\boldsymbol{X}_{62} = -\eta\boldsymbol{D}_f\boldsymbol{C}_\delta, \quad \boldsymbol{X}_{64} = -\eta\boldsymbol{D}_f\boldsymbol{D}_\delta,$$
$$\boldsymbol{X}_{63} = [\boldsymbol{J}_\delta \quad -\bar{\eta}\boldsymbol{D}_f\boldsymbol{D}_\delta],$$

and $\bar{\eta} = 1 - \eta$. *When the above inequality holds, the filter* Σ_f *in (8) with filter gains*

$$\boldsymbol{A}_f = \boldsymbol{U}^{-1}\boldsymbol{W}_a\boldsymbol{U}^{-\mathrm{T}}, \quad \boldsymbol{B}_f = \boldsymbol{U}^{-1}\boldsymbol{W}_b,$$
$$\boldsymbol{C}_f = \boldsymbol{W}_c\boldsymbol{U}^{-\mathrm{T}}, \qquad \boldsymbol{D}_f = \boldsymbol{D}_f \quad (19)$$

is a solution to the considered robust filtering problem, where $\boldsymbol{U}$ *is a nonsingular matrix satisfying* $\boldsymbol{U}\boldsymbol{U}^T = \boldsymbol{X} - \Phi$.

Proof: The proof is provided in Appendix.

By Lemma 1 and Schur complement, it is straightforward to rewrite (15) in Theorem 2 under the LMI framework. The resultant LMIs and the corresponding filter gains are presented in Theorem 3 below.

Theorem 3 *For a given* $\mu_e > 0$, *the filtering error dynamics* Σ_e *in (9) is exponentially mean-square admissible and satisfies (14) with all considered uncertainties, if there exist positive real scalars* $\{\varepsilon_a, \varepsilon_b, \varepsilon_c\}$, *matrices* $\boldsymbol{S} \in \mathcal{R}^{(n-r)\times n}$, $\boldsymbol{W}_a \in \mathcal{R}^{n\times n}$, $\boldsymbol{W}_b \in \mathcal{R}^{n\times p}$, $\boldsymbol{W}_c \in \mathcal{R}^{q\times n}$, $\boldsymbol{D}_f \in \mathcal{R}^{q\times p}$, *and positive definite matrices* $\{\boldsymbol{X}, \Phi, \boldsymbol{Q}\} \in \mathcal{R}^{n\times n}$, *such that the LMI*

$$\begin{bmatrix} \bar{\boldsymbol{M}}_{11} & \bar{\boldsymbol{M}}_{21}^{\mathrm{T}} \\ \bar{\boldsymbol{M}}_{21} & \bar{\boldsymbol{M}}_{22} \end{bmatrix} < \mathbf{0} \quad (20)$$

is satisfied, where

$$\bar{\boldsymbol{M}}_{11}=\begin{bmatrix}\boldsymbol{M}_{11} & * & * & * & * & * \\ \boldsymbol{0} & \boldsymbol{M}_{22} & * & * & * & * \\ \boldsymbol{M}_{31} & \boldsymbol{0} & \boldsymbol{M}_{33} & * & * & * \\ \boldsymbol{0} & \boldsymbol{M}_{42} & \boldsymbol{0} & \boldsymbol{M}_{44} & * & * \\ \boldsymbol{M}_{51} & \boldsymbol{M}_{52} & \boldsymbol{M}_{53} & \boldsymbol{M}_{54} & \boldsymbol{M}_{55} & * \\ \boldsymbol{M}_{61} & \boldsymbol{M}_{62} & \boldsymbol{M}_{63} & \boldsymbol{M}_{64} & \boldsymbol{0} & -\boldsymbol{I}_q\end{bmatrix} \quad (21)$$

$$\bar{\boldsymbol{M}}_{21}=\begin{bmatrix}\boldsymbol{M}_{71} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{M}_{75} & \boldsymbol{M}_{76} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{M}_{85} & \boldsymbol{M}_{86} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{M}_{95} & \boldsymbol{M}_{96}\end{bmatrix},$$

$\bar{\boldsymbol{M}}_{22}=diag(\boldsymbol{M}_{77},\boldsymbol{M}_{88},\boldsymbol{M}_{99})$,

and

$$\boldsymbol{M}_{11}=\begin{bmatrix}\begin{pmatrix}\boldsymbol{Q}-\boldsymbol{E}^{\mathrm{T}}\boldsymbol{X}\boldsymbol{E}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{S} \\ +\boldsymbol{S}^{\mathrm{T}}\boldsymbol{R}^{\mathrm{T}}\boldsymbol{A}+\varepsilon_a\boldsymbol{F}_x^{\mathrm{T}}\boldsymbol{F}_x\end{pmatrix} & * \\ (\Phi-\boldsymbol{X})\boldsymbol{E} & \Phi-\boldsymbol{X}\end{bmatrix},$$

$$\mathbf{M}_{22}=-\boldsymbol{Q}+\varepsilon_c\boldsymbol{F}_{\lambda x}^{\mathrm{T}}\boldsymbol{F}_{\lambda x},$$

$$\mathbf{M}_{31}=\begin{bmatrix}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{S}+\varepsilon_a\boldsymbol{F}_u^{\mathrm{T}}\boldsymbol{F}_x & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix},$$

$$\mathbf{M}_{33}=\begin{bmatrix}-2\mu_e^2\boldsymbol{I}_m+\varepsilon_a\boldsymbol{F}_u^{\mathrm{T}}\boldsymbol{F}_u & * \\ -\mu_e^2\boldsymbol{I}_m & -\mu_e^2\boldsymbol{I}_m+\varepsilon_b\boldsymbol{F}_u^{\mathrm{T}}\boldsymbol{F}_u\end{bmatrix}$$

$$\boldsymbol{M}_{42}=\varepsilon_c\boldsymbol{F}_u^{\mathrm{T}}\boldsymbol{F}_x,\quad \boldsymbol{M}_{44}=-\mu_e^2\boldsymbol{I}+\varepsilon_c\boldsymbol{F}_u^{\mathrm{T}}\boldsymbol{F}_u,$$

$$\mathbf{M}_{51}=\begin{bmatrix}\Phi\boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{X}\boldsymbol{A}+\bar{\eta}\boldsymbol{W}_b\boldsymbol{C} & \boldsymbol{W}_a\end{bmatrix}, \quad (22)$$

$$\mathbf{M}_{52}=\begin{bmatrix}\boldsymbol{0} \\ \eta\boldsymbol{W}_b\boldsymbol{C}\end{bmatrix},$$

$$\mathbf{M}_{53}=\begin{bmatrix}\Phi\boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{X}\boldsymbol{B}+\bar{\eta}\boldsymbol{W}_b\boldsymbol{D} & \bar{\eta}\boldsymbol{W}_b\boldsymbol{D}\end{bmatrix},$$

$$\mathbf{M}_{54}=\begin{bmatrix}\boldsymbol{0} \\ \eta\boldsymbol{W}_b\boldsymbol{D}\end{bmatrix},\quad \boldsymbol{M}_{55}=-\begin{bmatrix}\Phi & \Phi \\ \Phi & \boldsymbol{X}\end{bmatrix},$$

$$\boldsymbol{M}_{61}=[\boldsymbol{L}-\bar{\eta}\boldsymbol{D}_f\boldsymbol{C} \quad -\boldsymbol{W}_c],$$

$$\mathbf{M}_{62}=-\eta\boldsymbol{D}_f\boldsymbol{C},\quad \boldsymbol{M}_{64}=-\eta\boldsymbol{D}_f\boldsymbol{D},$$

$$\mathbf{M}_{63}=[\boldsymbol{J}-\bar{\eta}\boldsymbol{D}_f\boldsymbol{D} \quad -\bar{\eta}\boldsymbol{D}_f\boldsymbol{D}],$$

$$\mathbf{M}_{71}=[\boldsymbol{H}_x^{\mathrm{T}}\boldsymbol{R}\boldsymbol{S} \quad \boldsymbol{0}],$$

$$\mathbf{M}_{76}=\boldsymbol{H}_z^{\mathrm{T}}-\bar{\eta}\boldsymbol{H}_y^{\mathrm{T}}\boldsymbol{D}_f^{\mathrm{T}},$$

$$\mathbf{M}_{75}=[\boldsymbol{H}_x^{\mathrm{T}}\Phi \quad \boldsymbol{H}_x^{\mathrm{T}}\boldsymbol{X}-\bar{\eta}\boldsymbol{H}_y^{\mathrm{T}}\boldsymbol{W}_b^{\mathrm{T}}],$$

$$\mathbf{M}_{85}=[\boldsymbol{0} \quad \bar{\eta}\boldsymbol{H}_y^{\mathrm{T}}\boldsymbol{W}_b^{\mathrm{T}}],\ \boldsymbol{M}_{86}=-\bar{\eta}\boldsymbol{H}_y^{\mathrm{T}}\boldsymbol{D}_f^{\mathrm{T}},$$

$$\mathbf{M}_{95}=[\boldsymbol{0} \quad \eta\boldsymbol{H}_y^{\mathrm{T}}\boldsymbol{W}_b^{\mathrm{T}}],\ \mathbf{M}_{96}=-\eta\boldsymbol{H}_y^{\mathrm{T}}\boldsymbol{D}_f^{\mathrm{T}},$$

$$\boldsymbol{M}_{77}=-\varepsilon_a\boldsymbol{I}_{d1},\ \boldsymbol{M}_{88}=-\varepsilon_b\boldsymbol{I}_{d1},\ \boldsymbol{M}_{99}=-\varepsilon_c\boldsymbol{I}_{d1},$$

and $\bar{\eta}=1-\eta$. *When the above LMIs hold, the filter* Σ_f *in (8) with filter gains*

$$\boldsymbol{A}_f=(\boldsymbol{X}-\Phi)^{-1}\boldsymbol{W}_a,\quad \boldsymbol{B}_f=(\boldsymbol{X}-\Phi)^{-1}\boldsymbol{W}_b, \quad (23)$$
$$\boldsymbol{C}_f=\boldsymbol{W}_c,\qquad \boldsymbol{D}_f=\boldsymbol{D}_f$$

is a solution to the considered robust filtering problem.

Proof: By Lemma 1 and the Schur complement, it is easy to verify that (17) is equivalent to (20) with $\varepsilon_a,\varepsilon_b,\varepsilon_c>0$. The proof is omitted for brevity. The only part that must be proved here is the relationship between (19) and (23), which can be established via the transfer function matrix $\boldsymbol{G}_f(z)$ of the filter from $\boldsymbol{y}_c(k)$ to $\boldsymbol{z}_f(k)$.

$$\begin{aligned}\boldsymbol{G}_f(z)&=\boldsymbol{W}_c[z(\boldsymbol{U}\boldsymbol{U}^{\mathrm{T}})+\boldsymbol{W}_a]^{-1}\boldsymbol{W}_b+\boldsymbol{D}_f\\ &=\boldsymbol{W}_c[z(\boldsymbol{X}-\Phi)+\boldsymbol{W}_a]^{-1}\boldsymbol{W}_b+\boldsymbol{D}_f\\ &=\boldsymbol{W}_c[(\boldsymbol{X}-\Phi)(z\boldsymbol{I}-(\boldsymbol{X}-\Phi)^{-1}\boldsymbol{W}_a)]^{-1}\boldsymbol{W}_b+\boldsymbol{D}_f\\ &=\boldsymbol{W}_c(z\boldsymbol{I}-(\boldsymbol{X}-\Phi)^{-1}\boldsymbol{W}_a)^{-1}(\boldsymbol{X}-\Phi)^{-1}\boldsymbol{W}_b+\boldsymbol{D}_f.\end{aligned} \quad (24)$$

Remark 1 *Based on Theorem 3, the following convex optimization problem may be formulated to find the* H_∞ *optimal filter of the form (8) such that (14) is satisfied with the minimal* μ_e*:*

$$\min_{\mu_e^2,\varepsilon_a,\varepsilon_b,\varepsilon_c,\Phi,\boldsymbol{X},\boldsymbol{S},\boldsymbol{W}_a,\boldsymbol{W}_b,\boldsymbol{W}_c,\boldsymbol{Q},\boldsymbol{D}_f}\mu_e^2, \quad (25)$$

subject to the LMI (20), $\{\varepsilon_a,\varepsilon_b,\varepsilon_c\}>0$, $\mu_e^2>0$ *and* $\{\boldsymbol{Q},\boldsymbol{X},\Phi\}>\boldsymbol{0}$.

4. NUMERICAL EXAMPLE

In this section, an example is worked out to illustrate the proposed filter design method. Suppose matrices of the system Σ in (2) are as follows:

$$\boldsymbol{E}=\begin{bmatrix}1.2 & 3 & 1.5 \\ 0 & 3 & 1.5 \\ 1.2 & 0 & 0\end{bmatrix},\quad \boldsymbol{B}=\begin{bmatrix}-1 \\ 1 \\ 0.2\end{bmatrix},$$

$$\boldsymbol{A}=\begin{bmatrix}-0.204 & -0.060 & -0.092 \\ -0.208 & -0.336 & -0.208 \\ -0.180 & 0.228 & 0.848\end{bmatrix}, \quad (26)$$

$$\boldsymbol{C}=[\ 0.1 \quad -0.4 \quad 0.5\ ],\qquad \boldsymbol{D}=-0.6,$$

$$\boldsymbol{L}=[\ -1 \quad 0.6 \quad -0.5\ ],\quad \boldsymbol{J}=0.$$

The uncertainty matrices in (4) are

$$\boldsymbol{H}_x^{\mathrm{T}}=[\ 1 \quad 2 \quad 1\ ],\ \boldsymbol{H}_y=0.2,\ \boldsymbol{H}_z=0.3, \quad (27)$$
$$\boldsymbol{F}_x=[\ -0.2 \quad -0.4 \quad -0.2\ ],\ \boldsymbol{F}_u=1,$$

and $|\Delta|\le 1$. It is easy to verify that $(\boldsymbol{E},\boldsymbol{A}+\boldsymbol{H}_x\Delta\boldsymbol{F}_x)$ is an admissible pair, and $rank\boldsymbol{E}=2$.

Consider an unreliable transmission network (5) with $\eta=0.2$. The corresponding H_∞ optimal filter is designed by solving the convex optimization problem mentioned in *Remark 1*, which is implemented by the *MATLAB LMI Control Toolbox* [4]. The resulting optimal μ_e is 2.7666, and the filter gains in (23) are found to be

$$\boldsymbol{A}_f = \begin{bmatrix} 0.1404 & -0.0397 & -0.2907 \\ -0.1182 & 0.0370 & 0.4833 \\ 0.3107 & -0.1082 & -0.4774 \end{bmatrix},$$

$$\boldsymbol{B}_f^{\mathrm{T}} = [\ 0.0105 \ \ 1.2069 \ \ -1.0078\ ], \qquad (28)$$

$$\boldsymbol{C}_f = [\ -0.0903 \ \ 0.0032 \ \ 0.6919],$$

$$\boldsymbol{D}_f = -1.6506.$$

In addition, Fig. 2 shows the corresponding results for the optimal μ_e with respect to different expected values η. It appears that a better H_∞ performance is achieved when there are less missing-data.

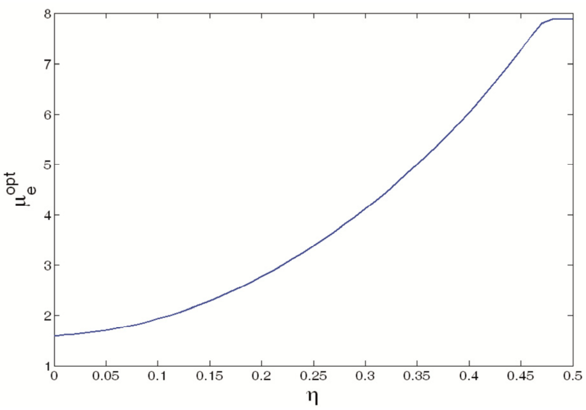

Fig. 2. Optimal μ_e with respect to different η

5. CONCLUSIONS

The networked filtering problem for a class of uncertain discrete-time singular systems with missing measurements due to unreliable transmission channels has been considered in this paper. It is easy to design a stable filter via a set of sufficient conditions under the LMI framework, such that the filtering error dynamic system is exponentially mean-square admissible and satisfies a prescribed level of H_∞ filtering performance. An optimal H_∞ filter can also be solved by the proposed convex optimization. One numerical example is provided to illustrate the effectiveness of the proposed approach. In the numerical example shown and other numerical experiments, it appears that a better H_∞ filtering performance is achieved when less packet-loss occurs.

6. APPENDIX

6.1. Proof of Theorem 1

The proof may be divided into three parts. The first part is to show that the filtering error dynamics Σ_{e} in (9) without any input $\boldsymbol{w}_e(\cdot)$ is regular and causal. We may rewrite the unforced filtering error system as

$$\Sigma_{\mathrm{eo}}: \boldsymbol{E}_e\boldsymbol{x}_e(k+1) = \widetilde{\boldsymbol{A}}_e\boldsymbol{x}_e(k) + \widetilde{\boldsymbol{A}}_{ed}\boldsymbol{x}_e(k-1), \qquad (29)$$

where $\widetilde{\boldsymbol{A}}_e$ and $\widetilde{\boldsymbol{A}}_{ed}$ are determined in (10). Based on the definition 3 of [16], the regularity and causality of Σ_{eo} can be ensured by the ones of

$$\Sigma_{\mathrm{ea}}: \widehat{\boldsymbol{E}}_e\widehat{\boldsymbol{x}}_e(k+1) = \widehat{\boldsymbol{A}}_e\widehat{\boldsymbol{x}}_e(k), \qquad (30)$$

where $\widehat{\boldsymbol{x}}_e^{\mathrm{T}}(k) = [\,\boldsymbol{x}_e^{\mathrm{T}}(k) \ \ \boldsymbol{x}_e^{\mathrm{T}}(k-1)\,]$,

$$\widehat{\boldsymbol{E}}_e = diag(\boldsymbol{E}_e, \boldsymbol{I}_{2n}), \quad \widehat{\boldsymbol{A}}_e = \begin{bmatrix} \widetilde{\boldsymbol{A}}_e & \widetilde{\boldsymbol{A}}_{ed} \\ \boldsymbol{I}_{2n} & \boldsymbol{0} \end{bmatrix}. \qquad (31)$$

Therefore, we establish the regularity and causality of the system Σ_{ea} via the corresponding matrices shown in (3), (10), (11), and

$$\begin{aligned} & det(z\widehat{\boldsymbol{E}}_e - \widehat{\boldsymbol{A}}_e) = \\ & (-1)^{2\cdot 2n}\cdot z^{2n}\cdot det(z\boldsymbol{E}_e - \widetilde{\boldsymbol{A}}_e - z^{-1}\widetilde{\boldsymbol{A}}_{ed}) \\ & = z^{2n}\cdot det\left(\begin{bmatrix} z\boldsymbol{E} - \boldsymbol{A}_\delta & \boldsymbol{0}(16) \\ \theta_{zk}\boldsymbol{B}_f\boldsymbol{C}_\delta & z\boldsymbol{I}_n - \boldsymbol{A}_f \end{bmatrix}\right) \\ & = z^{2n}\cdot det(z\boldsymbol{E} - \boldsymbol{A}_\delta)\cdot det(z\boldsymbol{I}_n - \boldsymbol{A}_f), \end{aligned} \qquad (32)$$

where $\theta_{zk} = -(\bar{\alpha}_k + z^{-1}\alpha_k)$. Since the pair $(\boldsymbol{E}, \boldsymbol{A}_\delta)$ in (2) is assumed to be admissible for all considered uncertainty $\delta\boldsymbol{A}$, and the filter gain $\boldsymbol{A}_f$ is Hurwitz, we get

$$det(z\boldsymbol{E} - \boldsymbol{A}_\delta) \neq 0, \quad det(z\boldsymbol{I}_n - \boldsymbol{A}_f) \neq 0, \qquad (33)$$

for sufficiently large z. There exists $z \in \mathcal{C}$ such that $det(z\widehat{\boldsymbol{E}}_e - \widehat{\boldsymbol{A}}_e) \neq 0$, which means that the pair $(\widehat{\boldsymbol{E}}_e, \widehat{\boldsymbol{A}}_e)$ is regular. On the other hand, based on (32),

$$deg(det(z\widehat{\boldsymbol{E}}_e - \widehat{\boldsymbol{A}}_e)) = 3n + rank\boldsymbol{E} = rank\widehat{\boldsymbol{E}}_e \quad (34)$$

The pair $(\widehat{\boldsymbol{E}}_e, \widehat{\boldsymbol{A}}_e)$ is causal. Therefore, the system Σ_{eo} is regular and causal.

Next, in order to show the the system Σ_{eo} is exponentially mean-square stable, we define a Lyapunov candidate as

$$\begin{aligned} V(k) = \ & \boldsymbol{x}_e^{\mathrm{T}}(k)\boldsymbol{E}_e^{\mathrm{T}}\boldsymbol{P}_e\boldsymbol{E}_e\boldsymbol{x}_e(k) + \\ & \boldsymbol{x}_e^{\mathrm{T}}(k-1)\boldsymbol{H}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{H}\boldsymbol{x}_e(k-1) \end{aligned} \qquad (35)$$

with $\boldsymbol{P}_e, \boldsymbol{Q} > \boldsymbol{0}$. Let F be the minimal σ-algebra generalized by $\{\boldsymbol{x}_f(i), 0 \leq i \leq k\}$. Via some straightforward algebraic manipulations, we have

$$\mathrm{E}\{V(k+1)|\mathcal{F}\} - V(k) = \boldsymbol{\xi}_e^{\mathrm{T}}(k)\Omega_e\boldsymbol{\xi}_e(k), \qquad (36)$$

where

$$\Omega_e = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix}, \qquad (37)$$

$\boldsymbol{\xi}_e^{\mathrm{T}}(k) = [\boldsymbol{x}_e^{\mathrm{T}}(k) \ \ \boldsymbol{x}_e^{\mathrm{T}}(k-1)\boldsymbol{H}^{\mathrm{T}}]$, and

$$\begin{aligned} \Omega_{11} &= \widetilde{\boldsymbol{A}}_e^{\mathrm{T}}\boldsymbol{P}_e\widetilde{\boldsymbol{A}}_e + \boldsymbol{H}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{H} - \boldsymbol{E}_e^{\mathrm{T}}\boldsymbol{P}_e\boldsymbol{E}_e, \\ \Omega_{21} &= \Omega_{12}^{\mathrm{T}} = \underline{\boldsymbol{A}}_{ed}^{\mathrm{T}}\boldsymbol{P}_e\widetilde{\boldsymbol{A}}_e, \\ \Omega_{22} &= \underline{\boldsymbol{A}}_{ed}^{\mathrm{T}}\boldsymbol{P}_e\underline{\boldsymbol{A}}_{ed} - \boldsymbol{Q}, \end{aligned} \qquad (38)$$

and $\underline{\boldsymbol{A}}_{ed} = \bar{\boldsymbol{A}}_{ed} + \delta\bar{\boldsymbol{A}}_{ed}$ is determined as in (13). It is not difficult to obtain from (15) and Schur complement that $\boldsymbol{\Omega_e} < \boldsymbol{0}$, which implies that

$$\begin{aligned} \mathrm{E}\{V(k+1)|\mathcal{F}\} - V(k) &= \boldsymbol{\xi}_e^{\mathrm{T}}(k)\Omega_e\boldsymbol{\xi}_e(k) \\ &\le -\lambda_{min}(-\Omega_e)\boldsymbol{\xi}_e^{\mathrm{T}}(k)\boldsymbol{\xi}_e(k) \\ &< -\beta\boldsymbol{\xi}_e^{\mathrm{T}}(k)\boldsymbol{\xi}_e(k), \end{aligned} \tag{39}$$

where

$$\begin{aligned} &0 < \beta < min\{\lambda_{min}(-\Omega_e), \phi\}, \\ &\phi = max\{\lambda_{max}(\boldsymbol{E}_e^{\mathrm{T}}\boldsymbol{P}_e\boldsymbol{E}_e), \lambda(\boldsymbol{Q})\} \neq 0, \end{aligned} \tag{40}$$

and $\lambda_{min/max}(\boldsymbol{M})$ denotes the minimal or maximal eigenvalue of the square matrix $\boldsymbol{M}$. By definition 1 and Lemma 1 of [18], we know that the system is exponentially mean-square stable.

At last, in order to show the filtering error $\boldsymbol{e}(k)$ satisfies the H_∞ performance (14), let

$$J_N = \sum_{k=0}^{N-1} \mathrm{E}\{\| \boldsymbol{e}(k) \|^2\} - \mu_e^2 \| \bar{\boldsymbol{w}}(k) \|^2. \tag{41}$$

For any nonzero $\bar{\boldsymbol{w}}(k) \in l_2[0,\infty)$ and zero initial conditions,

$$J_N = \sum_{k=0}^{N-1} \bar{J}_k + \mathrm{E}\{V(0)\} - \mathrm{E}\{V(N)\} \le \sum_{k=0}^{N-1} \bar{J}_k, \tag{42}$$

where

$$\begin{aligned} \bar{J}_k &= \mathrm{E}\{\| \boldsymbol{e}(k) \|^2\} - \mu_e^2 \| \bar{\boldsymbol{w}}(k) \|^2 \\ &\quad + \mathrm{E}\{V(k+1) - V(k)\} = \bar{\xi}^{\mathrm{T}}(k)\bar{\Omega}(\eta)\bar{\xi}(k), \end{aligned} \tag{43}$$

$\bar{\xi}^{\mathrm{T}}(k) = [\boldsymbol{x}_e^{\mathrm{T}}(k)\ \boldsymbol{x}_e^{\mathrm{T}}(k-1)\boldsymbol{H}^{\mathrm{T}}\ \boldsymbol{w}_e^{\mathrm{T}}(k)\ \boldsymbol{w}_e^{\mathrm{T}}(k-1)\boldsymbol{H}_b^{\mathrm{T}}\ ]$, and

$$\begin{aligned} \bar{\Omega}(\eta) &= \begin{bmatrix} \tilde{\boldsymbol{A}}_e^{\mathrm{T}}(\eta) \\ (\bar{\boldsymbol{A}}_{ed}^{\delta})^{\mathrm{T}}(\eta) \\ \tilde{\boldsymbol{B}}_e^{\mathrm{T}}(\eta) \\ (\bar{\boldsymbol{B}}_{ed}^{\delta})^{\mathrm{T}}(\eta) \end{bmatrix} \boldsymbol{P}_e \begin{bmatrix} \tilde{\boldsymbol{A}}_e^{\mathrm{T}}(\eta) \\ (\bar{\boldsymbol{A}}_{ed}^{\delta})^{\mathrm{T}}(\eta) \\ \tilde{\boldsymbol{B}}_e^{\mathrm{T}}(\eta) \\ (\bar{\boldsymbol{B}}_{ed}^{\delta})^{\mathrm{T}}(\eta) \end{bmatrix}^{\mathrm{T}} \\ &+ \begin{bmatrix} \tilde{\boldsymbol{C}}_e^{\mathrm{T}}(\eta) \\ (\bar{\boldsymbol{C}}_{ed}^{\delta})^{\mathrm{T}}(\eta) \\ \tilde{\boldsymbol{D}}_e^{\mathrm{T}}(\eta) \\ (\bar{\boldsymbol{D}}_{ed}^{\delta})^{\mathrm{T}}(\eta) \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{C}}_e^{\mathrm{T}}(\eta) \\ (\bar{\boldsymbol{C}}_{ed}^{\delta})^{\mathrm{T}}(\eta) \\ \tilde{\boldsymbol{D}}_e^{\mathrm{T}}(\eta) \\ (\bar{\boldsymbol{D}}_{ed}^{\delta})^{\mathrm{T}}(\eta) \end{bmatrix}^{\mathrm{T}} + \\ &diag(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{H} - \boldsymbol{E}_e^{\mathrm{T}}\boldsymbol{P}_e\boldsymbol{E}_e, -\boldsymbol{Q}, -\mu_e^2\boldsymbol{I}, -\mu_e^2\boldsymbol{I}) \\ &+ \begin{bmatrix} \bar{\boldsymbol{A}}^{\mathrm{T}} \\ \boldsymbol{0} \\ \bar{\boldsymbol{B}}^{\mathrm{T}} \\ \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{R}}^{\mathrm{T}} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}^{\mathrm{T}} + \begin{bmatrix} \bar{\boldsymbol{R}}^{\mathrm{T}} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{A}}^{\mathrm{T}} \\ \boldsymbol{0} \\ \bar{\boldsymbol{B}}^{\mathrm{T}} \\ \boldsymbol{0} \end{bmatrix}^{\mathrm{T}}. \end{aligned} \tag{44}$$

Note that the last two terms of (44) induced by an auxiliary equation

$$\boldsymbol{x}^{\mathrm{T}}(k+1)\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{x}(k) + \boldsymbol{x}(k)^{\mathrm{T}}\boldsymbol{S}\boldsymbol{R}^{\mathrm{T}}\boldsymbol{E}\boldsymbol{x}(k+1) = 0 \tag{45}$$

with $\boldsymbol{E}^{\mathrm{T}}\boldsymbol{R} = \boldsymbol{0}$ will not affect equality of (43) but improve the possible feasibility of the corresponding problem. It follows from (15) and by Schur complement that $\bar{\boldsymbol{\Omega}}(\eta) < \boldsymbol{0}$. It implies that $J_N < 0$ for all N, including for $N = \infty$. Therefore, the system satisfies the H_∞ performance in (14). This completes the proof.

6.2. Proof of Theorem 2

With $\boldsymbol{X} > \boldsymbol{0}$, $\Phi > \boldsymbol{0}$, inequality (17) implies its sub-matrices

$$-\begin{bmatrix} \Phi & \Phi \\ \Phi & \boldsymbol{X} \end{bmatrix} < \boldsymbol{0}, \tag{46}$$

which means that $\boldsymbol{X} - \Phi > \boldsymbol{0}$. Thus, $\boldsymbol{I} - \boldsymbol{X}\Phi^{-1}$ is nonsingular and there exist nonsingular matrices $\boldsymbol{U}$ and $\boldsymbol{V}$ such that $\boldsymbol{I} - \boldsymbol{X}\Phi^{-1} = \boldsymbol{U}\boldsymbol{V}^{\mathrm{T}}$. Let

$$\bar{\boldsymbol{P}} = \begin{bmatrix} \Phi^{-1} & \boldsymbol{I} \\ \boldsymbol{V}^{\mathrm{T}} & \boldsymbol{0} \end{bmatrix}, \qquad \hat{\boldsymbol{P}} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{X} \\ \boldsymbol{0} & \boldsymbol{U}^{\mathrm{T}} \end{bmatrix}. \tag{47}$$

Note that $\bar{\boldsymbol{P}}$ is nonsingular and therefore $\bar{\boldsymbol{P}}^{-1} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{V}^{-\mathrm{T}} \\ \boldsymbol{I} & -\Phi^{-1}\boldsymbol{V}^{-\mathrm{T}} \end{bmatrix}$. Define $\boldsymbol{P}_e = \hat{\boldsymbol{P}}\bar{\boldsymbol{P}}^{-1}$. By letting $\boldsymbol{U} = -\Phi\boldsymbol{V}$, $\boldsymbol{P}_e = \begin{bmatrix} \boldsymbol{X} & \boldsymbol{U} \\ \boldsymbol{U}^{\mathrm{T}} & \boldsymbol{I} \end{bmatrix}$. Moreover, under this arrangement $\boldsymbol{P}_e > \boldsymbol{0}$ because $\boldsymbol{X} - \boldsymbol{U}\boldsymbol{U}^{\mathrm{T}} = \boldsymbol{X} + \boldsymbol{U}\boldsymbol{V}^{\mathrm{T}}\Phi = \Phi > \boldsymbol{0}$.

Pre- and post-multiply (17) by $\boldsymbol{X}_a$ and $\boldsymbol{X}_a^{\mathrm{T}}$, respectively, with $\boldsymbol{U}\boldsymbol{U}^{\mathrm{T}} = \boldsymbol{X} - \Phi$, where

$$\boldsymbol{X}_a = diag(diag(\boldsymbol{I}, \boldsymbol{U}^{-1}), \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, diag(\Phi^{-1}, \boldsymbol{I}), \boldsymbol{I}). \tag{48}$$

Substituting (10), (13), (19), $\boldsymbol{U} = -\Phi\boldsymbol{V}$ and $\boldsymbol{P}_e = \hat{\boldsymbol{P}}\bar{\boldsymbol{P}}^{-1}$ to the resultant inequality sequentially, as well as pre- and post-multiplying by $\boldsymbol{X}_b^{\mathrm{T}}$ and $\boldsymbol{X}_b$, respectively, where

$$\boldsymbol{X}_b = diag(\boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, \bar{\boldsymbol{P}}^{-1}, \boldsymbol{I}), \tag{49}$$

result in (15). By Theorem 1, the filtering error dynamics in (9) is exponentially mean-square admissible, which implies the filter in (8) with gains in (19) is stable, and the H_∞ performance requirement (14) is satisfied for all admissible uncertainties.

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