An Analytic Closed-form Solution for Trajectory Generation on a Path along an Arc of a Circle

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ABSTRACT:
A polynomial trajectory is a time-traveled distance function used to describe trajectory of the robot. Optimal high-degree polynomial trajectories considering initial and the final velocity conditions besides the acceleration constraints are desired. In this paper, a trajectory optimization problem aiming travel maximum distance for a robot that follows an arc based path is formulated. Along the path, the robot requires observing initial and final zero velocity conditions as well as certain acceleration limits. A high-degree polynomial equation along the trajectory is proposed inside of the optimization problem. The closed-form solution of the problem had been obtained analytically. The solution includes the coefficients of the any high-degree trajectory polynomial equation where the coefficients are obtained in closed-form. Simulations several experiments show that the resulting high-degree trajectories satisfy the initial and final zero velocity conditions as well as acceleration constraint.

KEYWORDS: Trajectory Planning; Mobile Robot; Constrained Optimization; Analytic Solution.

1. INTRODUCTION

The task of describing the planar motion of a vehicle as a function of time stands on mathematical physics [1]. Based on physical laws and using definitions of physical quantities, the equation of motion can forms. Also, the boundary and initial value conditions can be settled. Then, a motion function for a vehicle can be obtained based on its dynamics, and the boundary and initial conditions [2].

Time planning of a path that a robot can follow is called it's trajectory planning, where it's an important task in motion planning for ground vehicles, flying vehicles or robots, unmanned aircraft, spacecraft and etc. An infinite number of time based trajectories reveals to traversing a given path segment. A limited number of these trajectories are appreciated to be tracked by satisfying some optimality conditions. Motion with maximum speed, maximums path traversal, spend minimum time, or a minimum turning radius are some of constraint that could be considered in the mobile robots optimal trajectory planning.

Optimal trajectories can be obtained from a time-optimal problem formulation such that the optimal motion is usually expressed as an optimization problem where the objective is to minimize the time of motion under the constraints. Therefore, there is need to develop methods to solve the trajectory optimization problems with some of mentioned constraints. In [3], a number of methods for trajectory optimization problems based on formulating and solving constrained nonlinear optimal control problems are reviewed. In many of trajectory optimization problems, time minimization and distance maximization of objective functions are often defined by taking kinematic constraints such as bounded velocities, accelerations and/or jerks into account. The goal is to find the optimum traveling distance, velocity and acceleration functions in the time domain according to the geometry of the path as well as the robot kinematics and dynamics.

Trajectory optimization for manipulator and mobile robots is studied in many papers. A manipulator trajectory planning with fifth-degree B-splines is presented in [4]. A mobile manipulator trajectory problem considering the torque and jerk constraint is investigated in [5]. Linear and circular path segments trajectory generation is studied in [6]. High-degree polynomial based S-curve trajectory planning under the constraints of minimum time and limited jerk-acceleration is studied in [7]. Some researches proposed the trajectories denoted by high-degree polynomials where a solution approach can be used to obtain the coefficients of these polynomials. Definitely, in polynomial trajectory optimization problems, determining the coefficients of these polynomials as closed-form is desired. However, closed-form solution is not simple to discover in many problems. In complicated problems, numerical solution methods can be used to solve the optimization problem. Solving trajectory problems with Particle Swarm Optimization [8], Genetic algorithm [9] and neural network [10] are only a number of numerical methods that are used in literature. Trajectory problems are investigated widely in robotic and engineering society, for example we could mention to [11-15]. Marine vehicles motion along curved paths in [16], unmanned aerial vehicles optimal trajectories in [17] and spacecraft maneuvers nonlinear trajectory optimization with path constraints in [18] are some engineering application of trajectories.

In polynomial trajectory problem, third-degree functions are widely used to interpolate the trajectory under continuous velocity and acceleration constraints. Third-degree polynomials are not smooth enough as they are unable to satisfy additional boundary constraints [19]. Use of high-degree polynomials requires additional coefficient computations. The main approaches of problem solving to find coefficients are analytically or numerical tracking of solutions. Evolutionary and numerical methods are sometimes unable to find an acceptable solution. Therefore, analytical approaches and closed-form solutions are desired always as the solutions can be determined by a number of simple mathematical operations.

In this paper, we proposed an any-order polynomial trajectory equation for describing motion of a point model mobile robot from its initial position to a goal position during a continuous set of time. The initial position and velocity profiles of the robot are known. Also, velocity profile is given in final position. Along the path, the robot requires to observe a certain acceleration limit. For describing such a motion, the considered trajectory function should be generating smooth position, velocity and acceleration profiles, while satisfy all mentioned constraints. We proposed a formulation of constrained trajectory planning problem where the solution can be obtained analytically. Proposed analytically solving procedure reaches to closed-form solution for coefficients of the considered high-degree polynomial trajectory. The closed-form coefficients generate an optimal trajectory function considering related path constraints. However, this problem are studied with similar condition with considering a third-order polynomial function in [20], or trajectory generation using any-order polynomial function for the straight path in [21], and (fourth-sixth)-order polynomials for manipulators in [22], apart of literature in this work we solved the problem for a case of two-term any-order polynomial function in the path of along an arc of a circle. In circular path, the constraint of limited acceleration is a nonlinear inequality that should be satisfied during of motion time. In other words, an optimization problem is formulated for describing of an arc path trajectory motion. The optimization problem is included of a two-term any-order polynomial function in the time domain under the initial and final velocity conditions as well as the acceleration constraint.

The rest of this paper organized as the sequel. The next section describes how we formulated optimization problem and then, the analytical solving method is presented. Simulation examples are provided in the third Section. The last section includes conclusions.

2. PROBLEM FORMULATION AND SOLUTION

2.1. Problem Formulation

Newtonian mechanics is an effective tool to describe motion equations in the form of a second order ordinary differential equation as,

$$\begin{cases} [s(t), \dfrac{\partial s}{\partial t}, \dfrac{\partial^2 s}{\partial t^2}] = 0 \\ s((0)), \dfrac{\partial s0}{\partial t}\Big|_{t=0}, s(t_f), \dfrac{\partial^2 s}{\partial t^2}\Big|_{t=t_f} \end{cases} \tag{1}$$

where s, t and t_f denote position, time variable and final time value, respectively. Note that first derivative and second derivative of position in time show velocity (v) and acceleration (a), respectively. Also, initial conditions as constant values should be given at $t = 0$. The solution s to the equation of motion describes the trajectory for all times $0<t<t_f$.

For a vehicle follows the path that stands on an arc of a circle, the trajectory equation could be described by position $s = s(t)$. Suppose that the vehicle travels from an initial angular position $\theta(0)$ to the final $\theta(t_f)$ along a circle with radius c. The length of arc or distance traveled by the vehicle is,

$$s(t_f) = (\theta(t_f) - \theta(0))c \tag{2}$$

In the planning of the vehicle trajectory in this path, we just have to determine the angular position over time. To this aim, assuming the vehicle moves from rest, we use a two-term any-order polynomial as below to describing its angular position.

$$\theta(t) = \lambda_1 t^n + \lambda_2 t^{n-1} \quad , \quad n \geq 3 \tag{3}$$

Now, the problem is to find λ_1, λ_2 in order to get an ptimum angular trajectory according to a formation of an optimization problem. Here, acceleration of the vehicle composed of centripetal (a_c) and tangential components (a_t) (see Fig. 1).

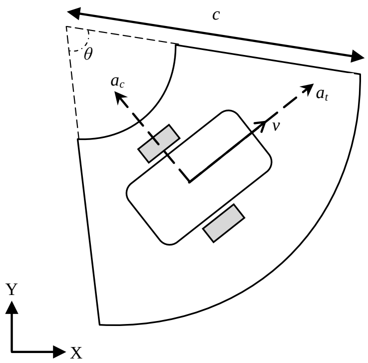


Fig. 1. The vehicle follows the path that stands on an arc of a circle.

Then, the limited acceleration constraint could be written as,

$$\left|\vec{a}_c + \vec{a}_t\right| < \Phi \tag{4}$$

where Φ is a constant value that we will use it to specify allowed maximum and minimum accelerations. Equation (4) shows a non-linear inequality constraint that satisfies limited acceleration condition in the case of trajectory at circular paths. Note that equation (4) could be written in the form of,

$$\sqrt{\left(\frac{v^2}{c}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2} < \Phi \qquad \text{or}$$

$$\frac{1}{c^2}\left(\frac{\partial s}{\partial t}\right)^4 + \left(\frac{\partial^2 s}{\partial t^2}\right)^2 < \Phi^2 \tag{5}$$

2.2. Solution

As can be seen, this paper discusses the problem of the motion based on time for a vehicle which has two degrees of freedoms in circular paths. In this subsection, optimal trajectory formulation and the solutions for the circular arc path segment trajectories are presented with the following assumptions. The robot is assumed as a point model. Also it is assumed that the robot is stationary and starts to move from origin on a circular arc path and finally it is stops gently at the end of the time or path. Aiming to have a safe navigation, it is assumed that acceleration / deceleration of the robot are limited.

In accordance with the assumptions mentioned above, the trajectory is desired to be planned satisfying the velocity boundary values and the acceleration constraint. In details, the trajectory optimization problem can be formulated with a high-degree polynomial subject to the velocity boundary condition and the acceleration constraint as following optimization problem,

$$\max_{\lambda_1, \lambda_2} \quad s(t_f) = c\left(\lambda_1 t_f^n + \lambda_2 t_f^{n-1}\right) \;, \quad n \geq 3 \tag{6}$$

subject to

$$v(t_f) = 0 \Rightarrow n\lambda_1 t_f^{n-1} + (n-1)\lambda_2 t_f^{n-2} = 0 \tag{6a}$$

$$|a| \leq \Phi \Rightarrow \left| \begin{matrix} \left\{ \begin{matrix} n(n-1)\lambda_1 t^{n-2} + \ldots \\ (n-1)(n-2)\lambda_2 t^{n-3} \end{matrix} \right\}^2 + \ldots \\ \left\{ n\lambda_1 t^{n-1} + (n-1)\lambda_2 t^{n-2} \right\}^4 \end{matrix} \right| < \Phi^2 c^{-2} \quad ,0 < t < t_f \tag{6b}$$

where, equation (6a) is a constraint which implies on zero velocity at the end of the trajectory and inequality (6b) is the limited acceleration constraint.

Finding λ_1 from equation (6a) and substituting in inequality (6b), the problem yields,

$$\max_{\lambda_1} \quad s(t_f) = \frac{1}{n} c\lambda_2 t_f^{n-1} \tag{7}$$

subject to

$$\left| \begin{matrix} \left\{ \dfrac{(n-1)\lambda_2}{t_f} \right\}^2 \ldots \\ \left\{ (n-2)t_f t^{n-3} - (n-1)t^{n-2} \right\}^2 + \ldots \\ \left\{ \dfrac{(n-1)\lambda_2}{t_f} \right\}^4 \left(t_f t^{n-2} - t^{n-1} \right)^4 \end{matrix} \right| < \Phi^2 c^{-2} \;, \quad 0 < t < t_f \tag{7a}$$

Differentiating inequality (6b) with respect to time, only one critical point obtained as,

$$t^* = \left(\frac{n-2}{n-1}\right) t_f$$

also, (6b) at time $t^{**} = t_f$ should be satisfied. Then, if inequality (7a) held at the obtained points, it would be held in all instances of interval $0<t<t_f$. Therefore, we can rewrite it as,

$$\left\{ \left(\frac{n-2}{n-1}\right)^{n-2} \lambda_2 t_f^{(n-2)} \right\}^4 < \Phi^2 c^{-2} \tag{7b}$$

$$\left\{ -(n-1)\lambda_2 t_f^{n-3} \right\}^4 < \Phi^2 c^{-2} \tag{7c}$$

for the times of t^* and t^{**}, respectively.

Inequalities (7b), and (7c) yields to,

$$\left(\Phi^{\frac{1}{2}} c^{\frac{-1}{2}}\right) \frac{-1}{t_f^{(n-2)}} \left(\frac{n-1}{n-2}\right)^{n-2} \ldots$$
$$< \lambda_2 < \left(\Phi^{\frac{1}{2}} c^{\frac{-1}{2}}\right) \frac{1}{t_f^{(n-2)}} \left(\frac{n-1}{n-2}\right)^{n-2} \tag{7d}$$

and

$$-\left(\Phi^{\frac{1}{2}}c^{\frac{-1}{2}}\right)\left(\frac{-1}{(n-1)}\right)\frac{1}{t_f^{n-3}}\ldots < \lambda_2 < \left(\Phi^{\frac{1}{2}}c^{\frac{-1}{2}}\right)\left(\frac{-1}{(n-1)}\right)\frac{1}{t_f^{n-3}} \tag{7e}$$

Maximum value of cost function in equation (7) can be obtained with maximum value of λ_2. Then the solution of the optimization problem is,

$$\lambda_2 = \min\left(\begin{matrix}\left(\Phi^{\frac{1}{2}}c^{\frac{-1}{2}}\right)\frac{1}{t_f^{(n-2)}}\left(\frac{n-1}{n-2}\right)^{n-2},\ldots \\ \left(\Phi^{\frac{1}{2}}c^{\frac{-1}{2}}\right)\left(\frac{-1}{(n-1)}\right)\frac{1}{t_f^{n-3}}\end{matrix}\right) \tag{8a}$$

From equation (6a), also we obtain λ_1,

$$\lambda_1 = -\frac{(n-1)}{nt_f}\min\left(\begin{matrix}\left(\Phi^{\frac{1}{2}}c^{\frac{-1}{2}}\right)\frac{1}{t_f^{(n-2)}}\left(\frac{n-1}{n-2}\right)^{n-2},\ldots \\ \left(\Phi^{\frac{1}{2}}c^{\frac{-1}{2}}\right)\left(\frac{-1}{(n-1)}\right)\frac{1}{t_f^{n-3}}\end{matrix}\right) \tag{8b}$$

Equations (8a) and (8b) are the closed-form solutions of λ_2 and λ_1, respectively. As a result, the maximum angular trajectory can be obtained by using the following equation,

$$\theta(t) = \left(-\frac{(n-1)}{nt_f}t^n + t^{n-1}\right)\min\ldots \left(\begin{matrix}\left(\Phi^{\frac{1}{2}}c^{\frac{-1}{2}}\right)\frac{1}{t_f^{(n-2)}}\left(\frac{n-1}{n-2}\right)^{n-2},\ldots \\ \left(\Phi^{\frac{1}{2}}c^{\frac{-1}{2}}\right)\left(\frac{-1}{(n-1)}\right)\frac{1}{t_f^{n-3}}\end{matrix}\right) \tag{9}$$

and the corresponding maximum distance is,

$$\max s(t_f) = \frac{1}{n}ct_f^{n-1}\min\left(\begin{matrix}\left(\Phi^{\frac{1}{2}}c^{\frac{-1}{2}}\right)\frac{1}{t_f^{(n-2)}}\left(\frac{n-1}{n-2}\right)^{n-2},\ldots \\ \left(\Phi^{\frac{1}{2}}c^{\frac{-1}{2}}\right)\left(\frac{-1}{(n-1)}\right)\frac{1}{t_f^{n-3}}\end{matrix}\right) \tag{10}$$

Substituting equation (9) into equation (2), the vehicle optimal trajectory can be obtained.

3. SIMULATION EXAMPLES

To illustrate the accuracy of closed-form solution that is described in the previous section, we present three simulation examples.

Example 1. Considering a two-term third-order polynomial trajectory function, the optimization problem is as follow,

$$\max_{\lambda_1,\lambda_2} \quad s(t_f) = c\left(\lambda_1 t_f^3 + \lambda_2 t_f^2\right) \tag{11}$$

subject to

$$v(t_f) = 0 \Rightarrow 3\lambda_1 t_f^2 + 2\lambda_2 t_f = 0 \tag{11a}$$

$$|a| \le \Phi \Rightarrow \ldots$$
$$\left|\{6\lambda_1 t + 2\lambda_2\}^2 + \{3\lambda_1 t^2 + 2\lambda_2 t\}^4\right| < \Phi^2 c^{-2} \quad ,0 < t < t_f \tag{11b}$$

Using presented closed-form solution, to have optimal trajectory with aim of maximum traversal, the values of λ_1 and λ_2 should be set as below,

$$\lambda_2 = \min\left\{\left|\frac{2\Phi^{\frac{1}{2}}c^{\frac{-1}{2}}}{t_f}\right|,\left|\frac{\Phi c^{-1}}{-2}\right|\right\}$$

and

$$\lambda_1 = -\left(\frac{2}{3t_f}\right)\min\left\{\left|\frac{2\Phi^{\frac{1}{2}}c^{\frac{-1}{2}}}{t_f}\right|,\left|\frac{\Phi c^{-1}}{-2}\right|\right\} \tag{12}$$

and the corresponding maximum distance trajectory is,

$$s(t) = \left[-\left(\frac{2}{3t_f}\right)t^3 + t^2\right]\min\left\{\left|\frac{2\Phi^{\frac{1}{2}}c^{\frac{-1}{2}}}{t_f}\right|,\left|\frac{\Phi c^{-1}}{-2}\right|\right\},\ 0 < t < t_f \tag{13a}$$

$$\max_{\lambda_1,\lambda_2} \quad s(t_f) = \left(\frac{1}{3}\right)t^2 \min\left\{\left|\frac{2\Phi^{\frac{1}{2}}c^{\frac{-1}{2}}}{t_f}\right|,\left|\frac{\Phi c^{-1}}{-2}\right|\right\},\ t = t_f \tag{13b}$$

Example 2. For a two-term fourth-order polynomial trajectory function, we have,

$$\max_{\lambda_1,\lambda_2} \quad s(t_f) = c\left(\lambda_1 t_f^4 + \lambda_2 t_f^3\right) \tag{14}$$

subject to

$$v(t_f) = 0 \Rightarrow 4\lambda_1 t_f^3 + 3\lambda_2 t_f^2 = 0 \tag{14a}$$

$$|a| \le \Phi \Rightarrow \left|\{12\lambda_1 t^2 + 6\lambda_2 t\}^2 + \{4\lambda_1 t^3 + 3\lambda_2 t^2\}^4\right| < \Phi^2 c^{-2}$$
$$0 < t < t_f \tag{14b}$$

The values of λ_1 and λ_2 should be set as below,

$$\lambda_2 = \min\left\{\left|\frac{9}{4}\frac{\Phi^{\frac{1}{2}}c^{\frac{-1}{2}}}{t_f^2}\right|,\left|\frac{\Phi c^{-1}}{-3t_f}\right|\right\}$$

and

$$\lambda_1 = -\left(\frac{3}{4t_f}\right)\min\left\{\left|\frac{9}{4}\frac{\Phi^{\frac{1}{2}}c^{\frac{-1}{2}}}{t_f^2}\right|,\left|\frac{\Phi c^{-1}}{-3t_f}\right|\right\} \tag{15}$$

and maximum distance trajectory is,

$$s(t) = \left[-\left(\frac{3}{4t_f}\right)t^4 + t^3\right]\min\left\{\left|\frac{9}{4}\frac{\Phi^{\frac{1}{2}}c^{\frac{-1}{2}}}{t_f^2}\right|,\left|\frac{\Phi c^{-1}}{-3t_f}\right|\right\}, \quad 0 < t < t_f \tag{16}$$

Example 3. A fifth-order polynomial trajectory,

$$\max_{\lambda_1,\lambda_2} \quad s(t_f)=c\left(\lambda_1 t_f^5+\lambda_2 t_f^4\right) \tag{17}$$

subject to

$$v(t_f)=0 \Rightarrow 5\lambda_1 t_f^4+4\lambda_2 t_f^3=0 \tag{17a}$$

$$|a|\le \Phi \Rightarrow \left|\{20\lambda_1 t^3+12\lambda_2 t^2\}^2+\{5\lambda_1 t^4+4\lambda_2 t^3\}^4\right| < \Phi^2 c^{-2}$$
$$,0<t<t_f \tag{17b}$$

For λ_1 and λ_2, we can get the below solutions,

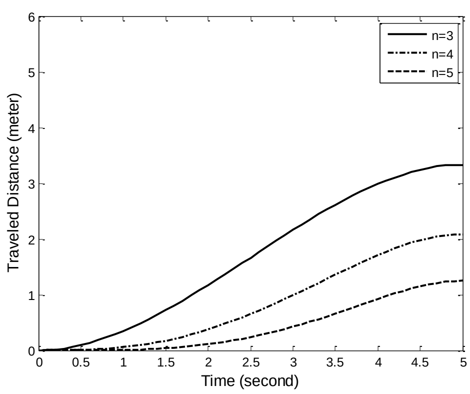
$$\lambda_2 = \min\left\{\left|\left(\frac{4}{3}\right)^3 \frac{\Phi^{\frac{1}{2}} c^{\frac{-1}{2}}}{t_f^3}\right|, \left|\frac{\Phi c^{-1}}{-4t_f^2}\right|\right\}$$

and $$\lambda_1 = -\left(\frac{4}{5t_f}\right)\min\left\{\left|\left(\frac{4}{3}\right)^3 \frac{\Phi^{\frac{1}{2}} c^{\frac{-1}{2}}}{t_f^3}\right|, \left|\frac{\Phi c^{-1}}{-4t_f^2}\right|\right\} \tag{18}$$

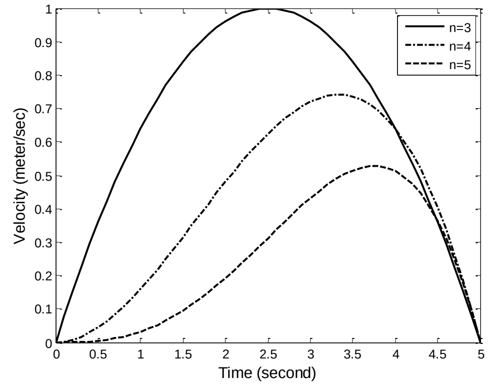
The optimal trajectory is,

$$s(t)=\left[-\left(\frac{4}{5t_f}\right)t^5+t^4\right]\min\left\{\left|\left(\frac{4}{3}\right)^3 \frac{\Phi^{\frac{1}{2}} c^{\frac{-1}{2}}}{t_f^3}\right|, \left|\frac{\Phi c^{-1}}{-4t_f^2}\right|\right\}$$
$$,0<t<t_f \tag{19}$$

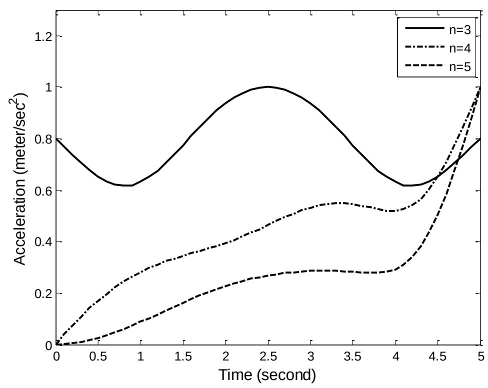
In the experiment motion simulations, setting t_f as 5 seconds, Φ as 1 (m/s^2) and c as 1 meter, the profiles of traveled distance, velocity and acceleration in the 3rd, 4th and 5th degree of the proposed trajectory polynomials are shown in Fig. 2. As can be seen in plots, travelled distance decreases as the degrees of the polynomials increase. The velocity boundary conditions are satisfied in the initial (t = 0) and the final times (t = t_f) as seen. The acceleration constraint is satisfied during the time interval [0, t_f] as shown in plots. Fig. 2 verifies the closed-form solutions as well as ensuring suitability of used the analytical problem solving approach in this paper



(a)



(b)



(c)

Fig. 2. Motion profile plots for 3rd, 4th and 5th degree of the polynomial trajectories, (a) traveled distances, (b) velocities, (c) accelerations.

4. CONCLUSION

We have proposed a formulation for trajectory planning problem denoted by high-degree polynomials. Formulation output is a time-distance optimal trajectory function for the vehicle in a circular path. Closed-form solution is presented to find the coefficients of trajectory polynomial that is proposed in the formulation. The formulation is taken into consideration under the velocity conditions as well as the acceleration constraint. Based on the obtained trajectories, it is easy to compute position, velocity, and acceleration at any time point throughout motion where these values can be used in the vehicle control system. By motion simulating of some examples, the accuracy of the used analytical approach is also ensured.

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