A New Approach for Solving Nonlinear Differential Equations with Poincare Map and Poincare Section

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ABSTRACT:
Solving many of the scientific problems in physics and engineering leads to differential equations, which in many cases no analytical answers can be found. The question to be considered is that, if the available numerical methods for solving deferential equations (that are all done by computers) are reliable.Is the Lipchitz validity assumption on differential equations with nonlinear dynamics true? What is the reason for the contradictory outcomes of solving a simple equation using numerical methods? Do the outcomes show the reality of the dynamic system? What is the acceptable replacement for the current methods?All the phenomena of the world have bifurcations, singularity, dissociation, behavioral changes and interaction; and today, science with assumptions like neglecting interactions and singularities, consider the systems as a continues model, although we are in need of a model in which we can solve the problem without inserting the changes in time approach to zero chain.In this article, accompanied by showing different and contradictory results –which are all wrong– numerical methods for solving a simple differential equation and comparing them with analytical method, we introduce Poincare as a substitution for overcoming this scientific derivation.In this article we solved a differential equation with common numerical methods in MATLAB, and showed that these methods produce conflicting outcomes, and then we solved it using Poincare. After showing the invalidity of common numerical methods and introduction of a simple decomposition method, we investigated Van der Pol equations using Poincare, and showed the fact that Poincare can simply show the system dynamics like a flashlight.

KEYWORDS: Poincare, Van der Pol oscillator, RungeKutta, Euler, Lipchitz

1. INTRODUCTION

A differential equation is an equation in which there is a function accompanied by its independent variant and derivations. The solution is found using either initiative or boundary conditions. A first order differential equation with initiative conditions is as follows:

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \quad (1)$$

We assume that function f is such that the differential equation (1) has a particular solution. Numerical methods for solving equation (1) is divided into two categories, one and multi-step methods. In one-step methods, we only use the point and the value and sometimes derivations value, however in multi-step methods $x_i$and some points before that are used [1-10]. In the presented methods in the following we have:

$$x_i = x_0 + ih \qquad i = 0,1,\dots,n \quad (2)$$

$$h = \frac{b-a}{n} \quad (3)$$

Taylor's method, Pth-Order

Assume that $y(x)$ is the precise solution for differential equation (1). In this method the approximate solution in x_{i+1} can be found as follows.

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!}y_i'' + \cdots + \frac{h_p}{p!}y_i^{(p)} + O(h^{p+1}) \quad (4)$$

To obtain an approximate of $y(x_{i+1})$ we consider the above expansion's rank up to p. In this situation local truncation error is $O(h^{p+1})$'s answer, moreover total Taylor Pth-Order error ، is $O(h^p)$. [1-5]

Euler's method

Assume that $y(x)$ is an answer for the differential equation (1), and assume that in $[x_i, x_{i+1}]$ interval we have:

$$f(x,y) \approx f(x_i, y_i)$$

With integrating $y' = f(x,y)$ in $[x_i, x_{i+1}]$ interval

and with error removal, we have:

$$y_{i+1} = y_i + hf(x_i, y_i) \quad (5)$$

Actually, Euler's method is Taylor's method considering the first two sentences. And in this method, local truncation error is $O(h^2)$. [6]

Modified Euler

In modified Euler method, we have:

$$\begin{cases} y_{i+1} = y_i + hf(x_i, y_i) \\ y_{i+1}^{(k)} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(k-1)}), k = 1,2, \dots \end{cases} \quad (6)$$

Usually this method is used for small value of $k = 2$. This method is also called implicit trapezoidal method [6].

Classic second order Runge Kutta method

In Classic Runge Kutta second order method, we use L1 and L2 for calculating the value of y(x) in the next point. The method is as follows: [11]

$$\begin{cases} L_1 = hf(x_i, y_i) \\ L_1 = hf(x_i + h, y_i + L_1) \\ y_{i+1} = y_i + 0.5(L_1 + L_2) \end{cases} \quad (7)$$

2. INVESTIGATING THE ACCURACY OF NUMERICAL

In economic, technologic or natural systems, we need to know the system's behavior in definite times, for instance the quantity of rainfall in a region, economic profits and etc. Basically systems are defined as events which in many cases do not have any chain relations with other point, either before or after them; but there is annular relation in which the role of cause and effect is interchangeable, and as in cybernetics the role of actor and audience is under change. It is where the weakness and problem within continues functions and ODE1 solutions are revealed and shown precisely to be wrong.

The question under consideration is that is the present numerical methods for solving differential equations –which are all done by computer and basically we do not have an analytical solution for them- are reliable? Is the assumption of "Lipchitz rules are valid about equations" and numerical methods reliable? To investigate the accuracy of answers, we subjected the $\dot{x} = \frac{1}{\sin(x)}$ equation to under close scrutiny by solving it using different numerical methods and MATLAB.

$$\dot{x} = \frac{1}{\sin(x)} \quad (8)$$

$$\frac{dx}{dt} = \frac{1}{\sin(x)} \rightarrow \int_{x_0}^{x} \sin(\gamma).d\gamma = \int_{t_0}^{t} d\theta \rightarrow -\cos(\gamma)|_{x_0}^{x}$$
$$= \theta|_{t_0}^{t} \rightarrow \cos(x) - \cos(x_0)$$
$$= -(t - t_0)$$
$$\rightarrow x = \cos^{-1}(-(t - t_0) + \cos(x_0)) \quad (9)$$

In fig.1, the outcome for the differential equation is illustrated (9). It should be mentioned that $\cos^{-1}(\theta)$ for θ out of [-1, 1] will be complex and is shown with dotted line in Fig .1.

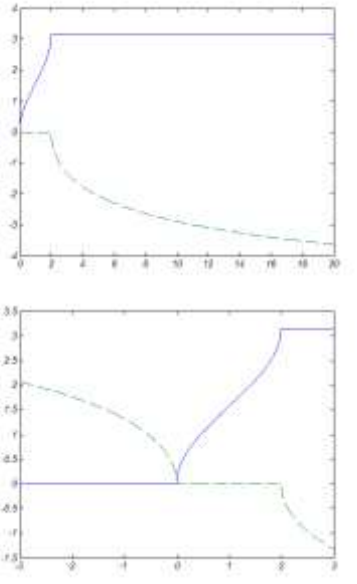


Fig. 1. Analytical outcome for the equation (8) complex part on dashed line.a. Horizontal axis is [0,20], b. Horizontal axis is [-3,3]

Then with SIMULINK in MATLAB using present numerical methods we solved the differential equation and drew the real part of their answer, after that we compared them with each other. In Fig 2 the simulated SIMULINK model of the differential equation can be seen and the other answers due to method change are charted in Fig.3.

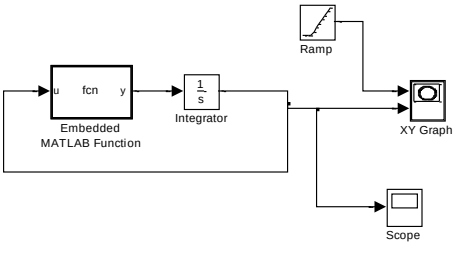
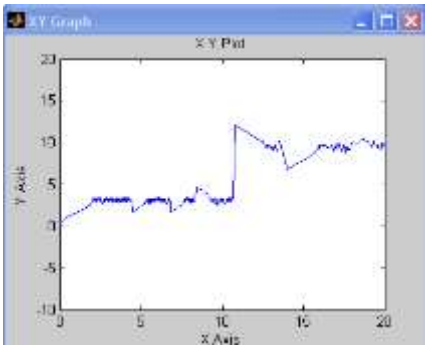
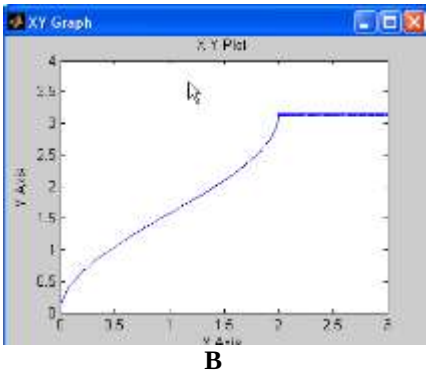


Fig. 2 SIMULINK Model for Equation(8)

[1]Ordinary Differential Equations



a



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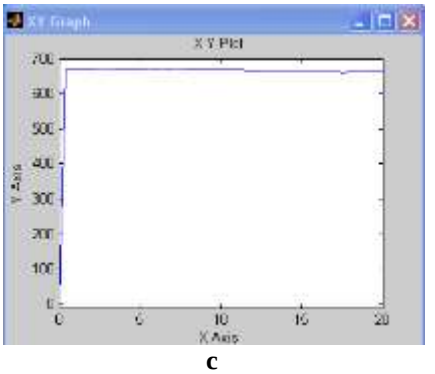

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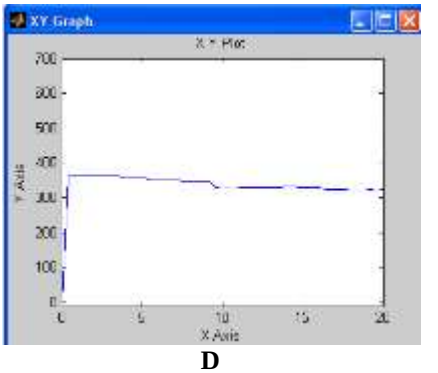

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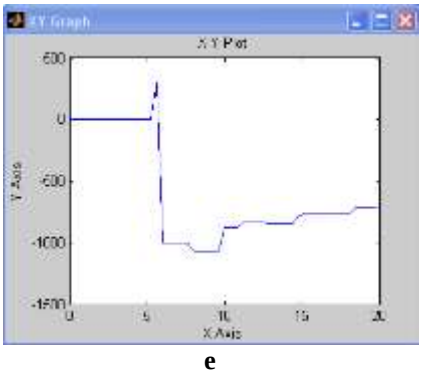

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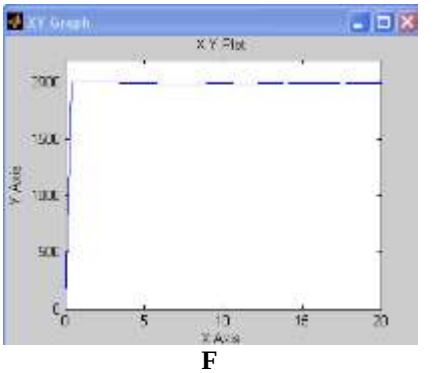

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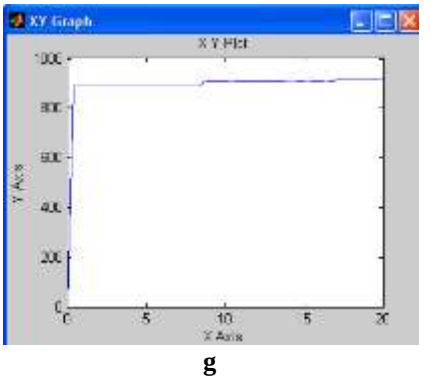
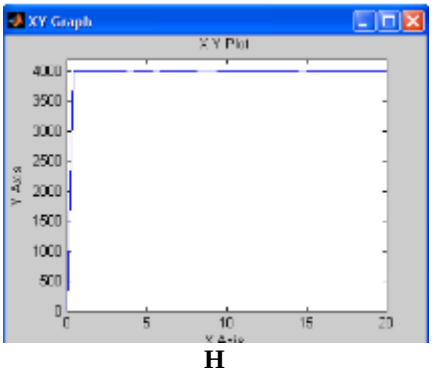


g



H

Fig.3 Different Numerical Analisys of Equation 8

a. Variable-step Dormand – Prince method (ode45) **b.** Variable-step Bogacki – shampinemethod (ode23) **c.** fixed-step Runge – Kutta method (ode4) d. fixed-step Dormand – Prince method (ode5) **e.** fixed-step extrapolation method (ode14x) **f.** fixed-step Heun method (ode2) **g.** fixed-step Bogacki – shampine method (ode3) **h.** fixed-step Euler method (ode1)

3. SUGGESTED METHOD

In section 2 it was shown that if Lipchitz rules are not valid for an equation, present numerical methods cannot be used for finding the correct answer. Continues models of differential equations occasionally either don not have analytical solutions or are difficult to solve. This problem in the cases in which a natural and especially biological system is presented, is of the main importance. Moreover, in many biological and natural phenomena, the essence of the real system is discontinues and therefore applying continues differential equations for modeling is not proper.

Hence, differential equations discretization and solving difference equations instead of differential equations are considered important. There are numerous methods for discretization and numerically solving a differential equation, yet facing instability in equivalent discrete models of the continues models is common. This instability can be caused due to various reasons. First of all when the grade of discrete difference equations are more than the grade of continues differential equations, fixed points of the discrete model are no more equal to continues points and this leads to unwanted instability. Secondly, in some discretization methods, the discrete difference equation does not follow the constraints of differential equation, like instability of energy, univocally, being bounded and being positive.

The solution for this problem is the application of a more precise discretization approach in which the time step is not limited to a fixed value, as it is in real discrete phenomena like Poincare cut in pendulum movement, Poincare cut points in electrical circuits in oscillating and semi-oscillating and chaotic behaviors. From the modeling point of view, time delay which is equivalent to delay, models the dependency of the system's outcome on the previous points or in other words with memory system.

From the theoretical point of view, the existence of a variable time step is an obstacle for instability in the answer of discrete difference equation which is equivalent to continuous differential equation. This method is named Nonstandard Finite Difference discretization method. Easily it can be shown that the discrete equivalent of a continuous system is far distinct from what is Euler's method outcome. The differential equation $\frac{du}{dt} = -\lambda u\,, u(t_0) = u_0$ has a exponential response in the form of $u(t) = u_0 e^{-\lambda(t-t_0)}$.

The discrete equivalent of the above equation using Euler's method is as follows:

$$\frac{u_{k+1}-u_k}{h} = -\lambda u_k \qquad (10)$$

Theory:

One order ordinary Differential Equation

$$\frac{du}{dt} = f(u,t,\lambda), \qquad u(t_0) = u_0$$

Answer is: $u(t+h) = \phi[\lambda, u(t), t, t+h]$

Variables changes are:

$t_0 \to t_k, \qquad t \to t_{k+1}, \qquad u_0 \to u_k, \qquad u(t) \to u_{k+1}$

discrete equivalent of differential Equation is:

$$u_{k+1} = \phi(\lambda, u_k, t_k, t_{k+h})$$

Considering the above mentioned theory, discrete equivalent of this difference equation response will be like this:

$u_{k+1} = u_k e^{-\lambda h}$

This can be rewritten as in below;

$$\to \frac{\frac{u_{k+1}-u_k}{\lambda}}{(\frac{1-u_k e^{-\lambda h}}{\lambda})} = -\lambda u_k \qquad (11)$$

As it is obviously seen, the discrete equivalent of this differential equation which was obtained analytically is conflicting with the discrete equivalent with was calculated using Euler's method.

And we can see, the numerical solution which was obtained using Euler's method is unstable for $h > 1$. The reason of this instability is the contradiction mentioned between differential and difference equation. [12]

Generally, the discrete equivalent of a continuous equation in the form of $\dot{X} = F(X)$, $X = (x_1, x_2, \dots)$ is far more complex that it can be estimated using Euler's method. Actually it can be rewritten as it follows:

$$\frac{X_{k+1} - \psi X_k}{\phi} = F(X_k)$$

In Euler's method $\dot{X} \to \frac{X_{k+1}-X_k}{h}$, however a more precise estimation should be $\dot{X} = \frac{X_{k+1}-\psi X_k}{\phi}$. h is the time step, then ϕ and ψ would be functions dependent on hin which generally the following conditions are true:

$\psi = 1 + O(h), \quad \phi = h + O(h^2)$

In these functions, often $\psi = 1$ and ϕ is chosen as follows:

$$\phi(h, R^*) = \frac{1 - e^{-R^* h}}{R^*}$$

For determining the value of R^*, first of all, the fixed points of differential equation have to be determined. Generally$\dot{X} = F(X)$, if I has a fixed point as $\{x_i; i = 1,2, \dots, I\}$:

$$R_i \equiv \left.\frac{dF}{dX}\right|_{X=x_i} \to R^* \equiv \max\{|R_i|; i = 1,2, \dots, I\}$$

In a dynamic system, if the independent variable is time, then according to the definitions given, the unit for R_i would be reverse time and time scales can be defined for the system as below:

$$T_i \equiv \frac{1}{R_i};\quad i = 1,2,\dots,I;\quad T^* = \frac{1}{R^*}$$

Hence T^* is the smallest time scale of the system and it can be shown that:

$$0 < \phi(h, R^*) < \frac{1}{R^*} \rightarrow 0 < \phi(h, T^*) < T^*$$

In definition of the physical concept of function ϕ, it can be said that ϕ is the normalized or scale-changed form of time step and its value is never larger that the smallest value of time scale. In other words the time step is efficient. Forasmuch as many of the instability problems of the numerical responses are due to it that value of time step is larger than a special amount, this method for choosing ϕ prevents such instabilities. Due to the normalization method applied, even if time step h would be very large, the efficient time step ϕ will not be larger than the allowed.

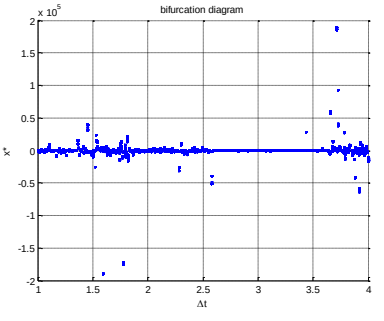
4. IMPLEMENTATION OF THE METHOD ON DIFFERENTIAL EQUATIONS MENTIONED

First of all we discretize the differential equation 8:

$$\dot{x} = \frac{1}{\sin(x)}$$

$$\rightarrow \frac{x_{k+1} - x_k}{\Delta t} = \frac{1}{\sin(x_k)}$$

$$\rightarrow x_{k+1} = \frac{\Delta t}{\sin(x_k)} + x_k \qquad (12)$$



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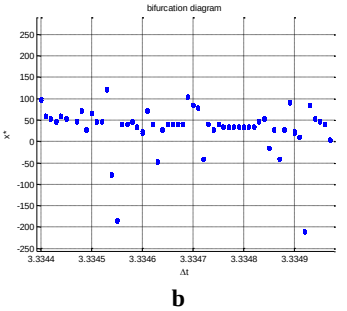


b

Fig.3 bifurcation Diagram of equation 8 for Δt variable, **a.**Δt Variation between 1 to 4 , **b.**Δt Variation between 3.3344 to 3.3349

In equations like the ones above which are not univocal, we use bifurcation chart for determining the direction of Δt for changes in Δt and obtaining the sought for behavior for the desiredΔt.

As it can be seen in the bifurcation chart, the differential equation for different value of Δt the final value of x is led to two amounts which one is due to the response of the real section and the other is due to imaginary section. Hence for this particular problem we put$\Delta t = 3.334$, then we have:

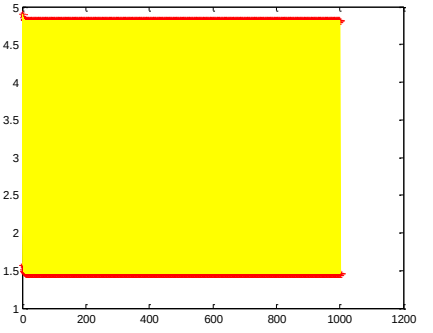


Fig. 4: Equation 8 for $\Delta t = 3.3345$

I.e. it is exactly led to two values; of course the final value is different from the final value of the analytical solution but with averaging the two responses, we can obtain a response similar to the analytical response which is not necessarily correct.

Recursive chart

Recursive chart or Poincare chat of the differential equation is given in the figure below and as we can obviously see two points in the extremity of this differential equation (what all methods based on fatalistic attitude of $\Delta t \rightarrow 0$ are helpless to describe)

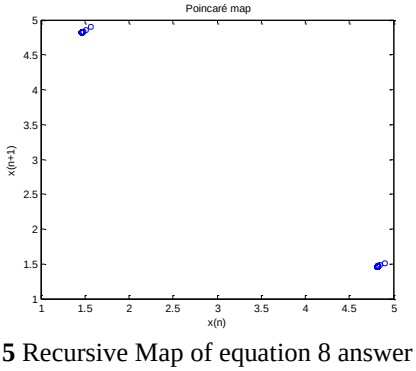


Fig. 5 Recursive Map of equation 8 answer with Suggested method

So far we talked about banishment of numerical methods but not about vindication of Poincare and that how Poincare can put an end to this scientific deviation which needs more and clearer examples. In the following we will take Van der Pol equation into consideration.

Van der Pol equation is a differential equation defined as follows: [13-18]

$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu(1 - x^2)y - x \end{cases}$$

Can we trust the outcome of numerical method

solution? Considering the abovementioned, numerical answers are not reliable.

Is it possible to recognize from the equation whether it has limit cycle? In the following we will find out that the answer is negative, since this equation has a fixed point at (0,0), in the study of system's stability we find out that:

$$\lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$$

$$\begin{cases} |\mu| > 2 \rightarrow \begin{cases} \mu > 2 \\ \Longrightarrow \lambda_{1,2} > 0 \ and \ Real \end{cases} \\ \qquad\qquad \mu < -2 \\ \qquad\qquad \Longrightarrow \lambda_{1,2} < 0 \ and \ Real \\ |\mu| < 2 \rightarrow \begin{cases} 0 < \mu < 2 \\ \Longrightarrow \lambda_{1,2} \text{ Complex, and Real}(\lambda_{1,2}) > 0 \end{cases} \\ \qquad\qquad -2 < \mu < 0 \\ \qquad\qquad \Longrightarrow \lambda_{1,2} \text{Complex and Real}(\lambda_{1,2}) < 0 \\ |\mu| = 2 \ \rightarrow \ \lambda_{1,2} = 1 \end{cases}$$

As it can be seen, the stability state of the system is dependent on μ.

Table 1. System Behavior

System behavior	μ value
Attractor	$\mu < -2$
Circular Attractor	$-2 < \mu < 0$
Circular repellent	$0 < \mu < 2$
radial repellent	$\mu > 2$

Hence, the system's behavior can be predicted as follows:

The points -2, 0, 2 are the system's bifurcation points, the points in which the system's behavior is changed in, leads to a change in extremity of the system, i.e. system's dynamic. The results of system's simulation shows that since $\mu < 0$, system's behavior is as we desire, but when $\mu > 0$, system's behavior does not follow the above table, and the limit cycle behavior appears. The point $\mu = 0$ in which the system's behavior changes from stable to unstable is named Hopf-Bifurcation. In references with the appearance of this point, the probability of limit cycle is strengthened, the question is: how should this problem be solved?

In order to solve this problem we apply the presented method in the previous example

$$\begin{cases} \dot{x} = \frac{x_{k+1} - x_k}{\Delta t} \xrightarrow{\Delta t=1} x_{k+1} - x_k = y_k \\ \dot{y} = \frac{y_{k+1} - y_k}{\Delta t} \xrightarrow{\Delta t=1} y_{k+1} - y_k = \mu(1 - {x_k}^2)y_k - x_k \end{cases}$$

The interpretation of each obtained shape in Poincare space is also of main importance. Fig 7 shows that if Poincare cut is done with the line x=0, points are getting further, actually the recursive equation 13 becomes the relation 14, and since =0.9μ, it will

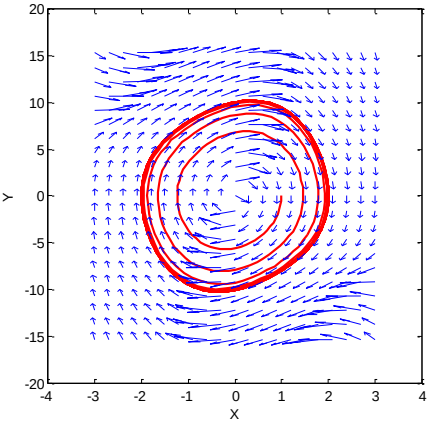
$$\begin{cases} x_{k+1} = y_k + x_k \\ y_{k+1} = \mu(1 - {x_k}^2)y_k - x_k + y_k \end{cases}$$

Now we want to use $x = \alpha y + \beta$ as Poincare cut and investigate the system's behavior.

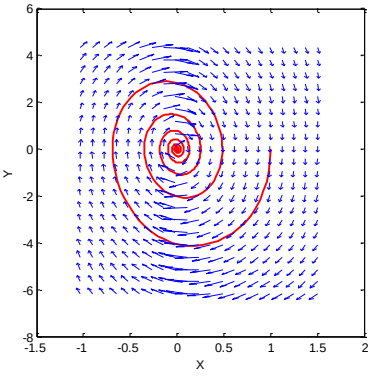
$$y_{k+1} = \mu(1 - (\alpha y_k + \beta)^2)y_k - (\alpha y_k + \beta) + y_k \quad (13)$$

In this recursive equation for a definite μ, we can gain desired values for α and β which have valuableinformation about Van der Pol equation trajectory, and now an unsolvable problem from numerical methods and analytical methods -based on fatalistic attitude- points of view is changed to an entirely solvable one.

For $\mu = 0.9$, set of points obtained from Poincare cut are illustrated in fig 7, no 1 and for tracking the changes in y_k in Poincare cut, the value of y_k in each sequence is also shown in Fig 7, no 2.



$\mu = 1$



$\mu = -1$

Fig.6 Van der Pol solution with Runga Kutta Method for $\mu = \pm 1$

increase with the increase of +1=1.9μ and as it can be seen, the slope of the line is constant.

$$y_{k+1} = \mu y_k + y_k = (\mu + 1)y_k \quad (15)$$

In the fig 7, the phase changes in set of points of Poincare cut in one hand, and the variations in values

on the other hand shows that if the Poincare cut happens with the line $x = 0.5y$, at first the distant between lines is considerable, and gradually it lowers and becomes fixed in a point. In the technique applied, actually, the line $x = \alpha y + \beta$ acts as a flashlight, which lightens the dark absorption bed as in set of bright points, this is very similar to what happens in stereoscopy (fig 8). In fig 9 the Poincare cut of $x = -y$ for more repetitions is illustrated and the result is sinus topology completion in Poincare space.

Fig.8: Stereoscopy pictures

Actually by the line $x = \alpha y + \beta$ we can as in stereoscopy pictures investigate the Van der Pol equations with high precision and without errors.

In the following using exponential scale, the geometrical changes of Poincare cut phase space $x = \alpha y$ can be illustrated more precisely.

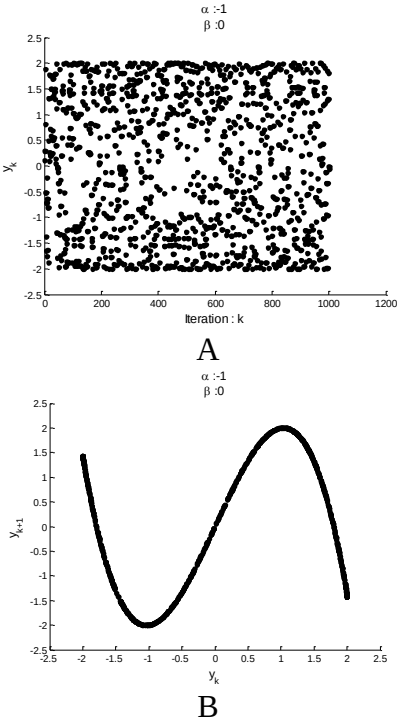


Fig.9 Phase Plane of Poincare Map for $x = -y$ with more iterations

Exponential scale have been applied in figures 10 and 11, so that big changes have less influence on small changes and that we can track the topology changes better. In fig 8, Poincare cut phase space in case of zero slope and zero y-intercept is shown. In fig 10, the slope is variable from negative with big absolute values to small absolute values and the Poincare cut phase space is drawn.

In fig 11 also, Poincare cut phase space in the case of positive slipe and zero y-intercept, is shown. In fig 9 slope changes from small amounts to big amounts and the Poincare cut phase is drawn.

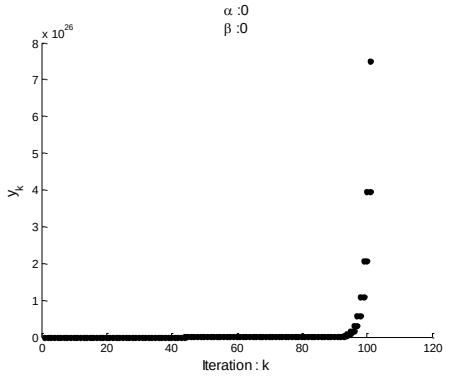
5. CONCLUSION

As it can be seen in the figures in table 1, due to singularity, the derivation of equation 8 is not defined in $x = k\pi$, and the answer of different numerical methods becomes distinctly different from the real answer after the singular point and since f(x) does not have Lipschitz conditions, numerical solution of the equation with assumption of $\Delta t \to 0$ has false response.

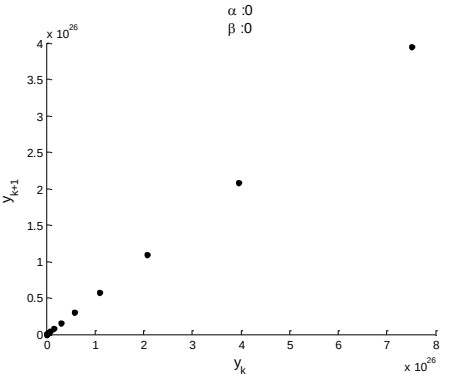
This method can be used in numerical solving differential equations, continuous equations discretization, and even finding Poincare cut from the differential equation model. As we know, Poincare cut gives us valuable information regarding to the dynamics and behavior of the system. What is of main importance is to find a way to identify Poincare cut points directly from the differential equation before solving it.

For instance, with finding Poincare cut points from the differential equation model, the existence of limit cycle and absorptions related to semi periodic and even chaotic behaviors can be recognized before solving the model and finding the trajectory. Considering variable time intervals in finding Poincare cut points is one of the facts that have been neglected even in famous equations such as logistic equation. But variable time interval, as mentioned, is a fact that exists in all natural phenomena and causes the model to be more physically tangible and closer to reality.

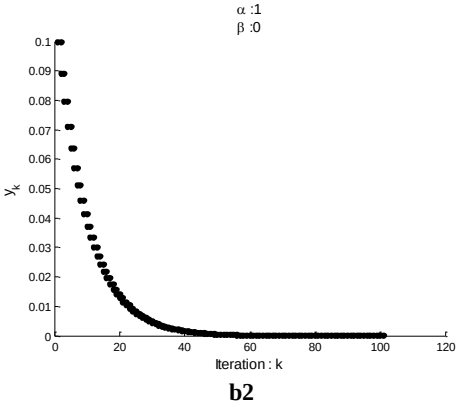
Actually, Poincare cut shows the deep down of the system's dynamic, and with adequate knowledge about it, we can conclude the whole system's dynamics without using Lipchitz theories.



a2



a1



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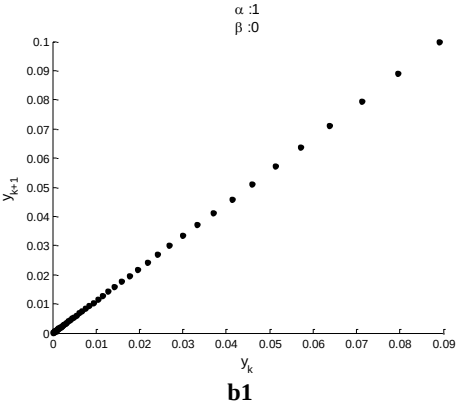

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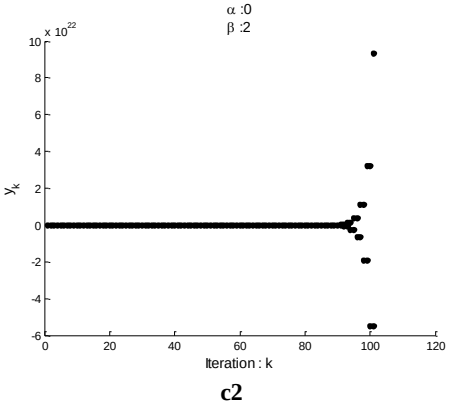

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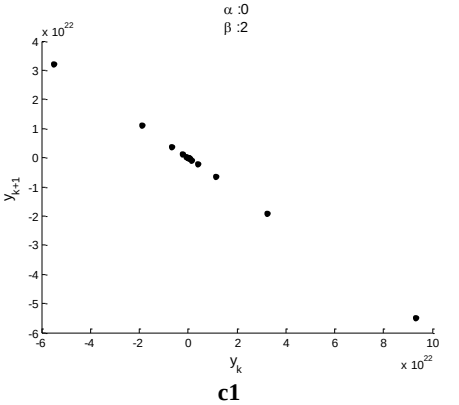

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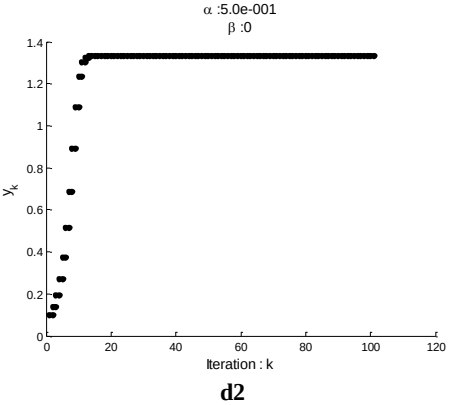
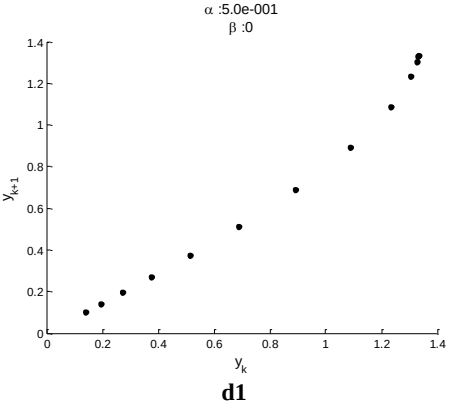


d2



d1

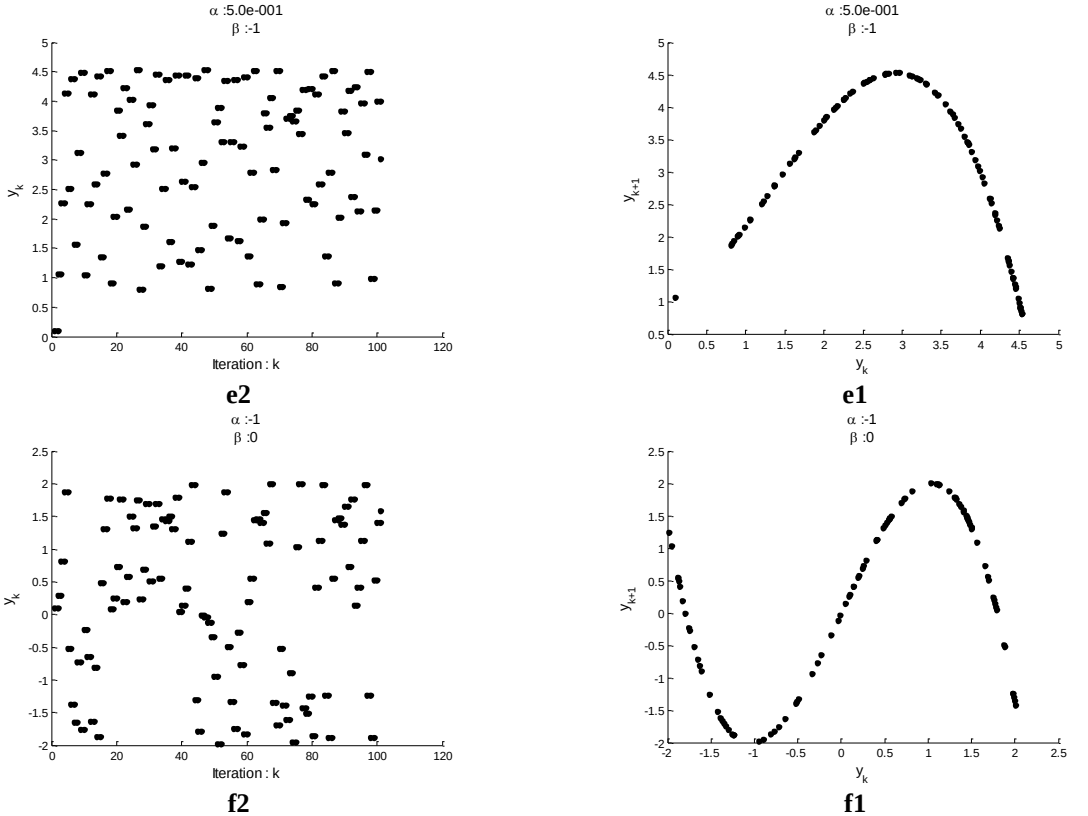
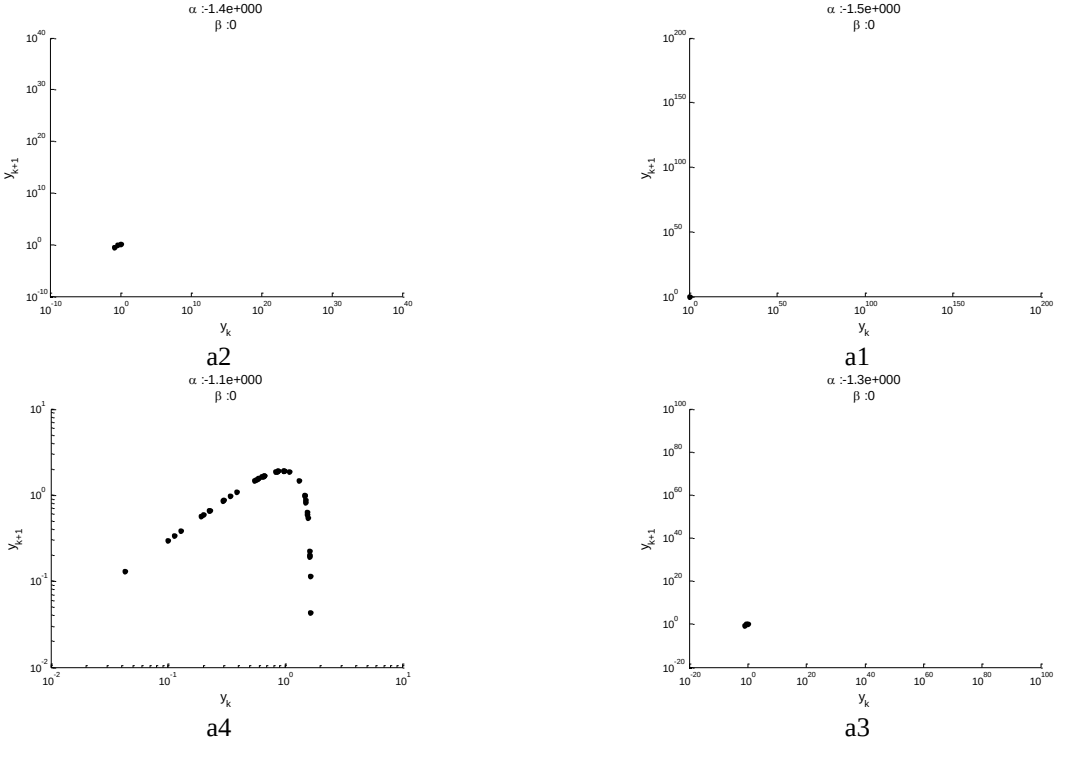


Fig.9 Phase Plane of Van der Pol Poincare Map when $\mu = 0.9$



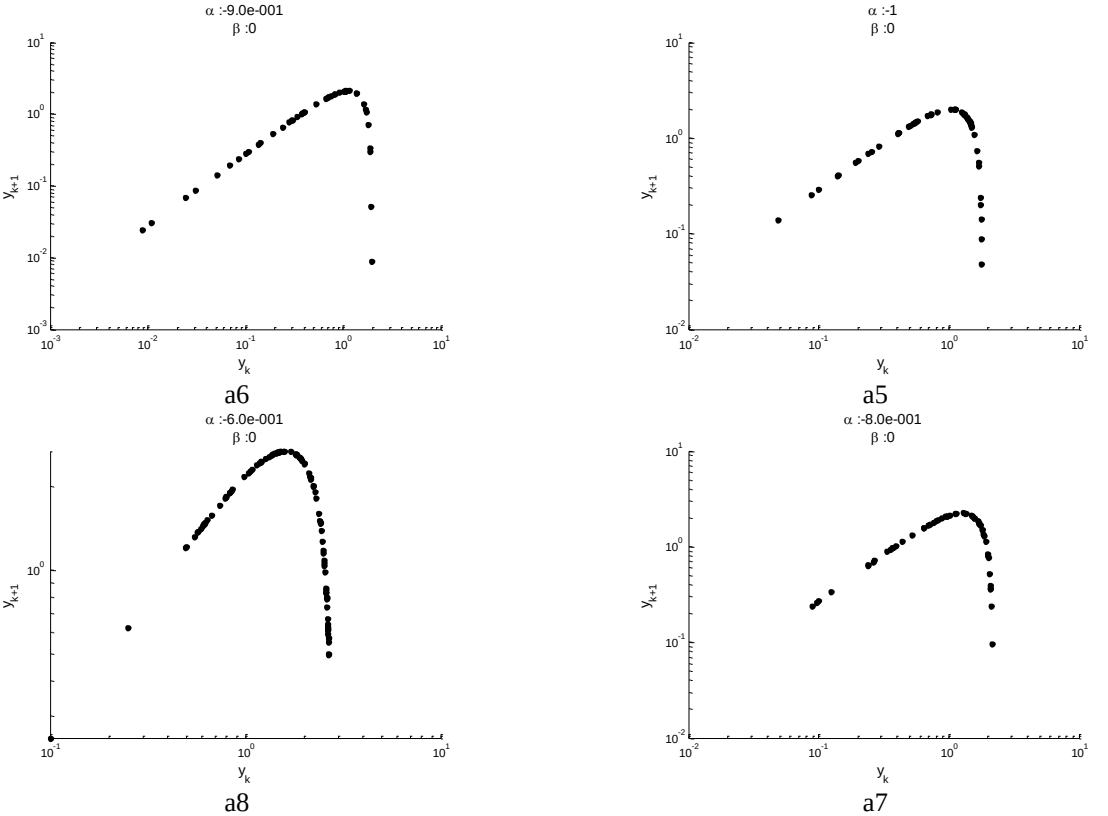
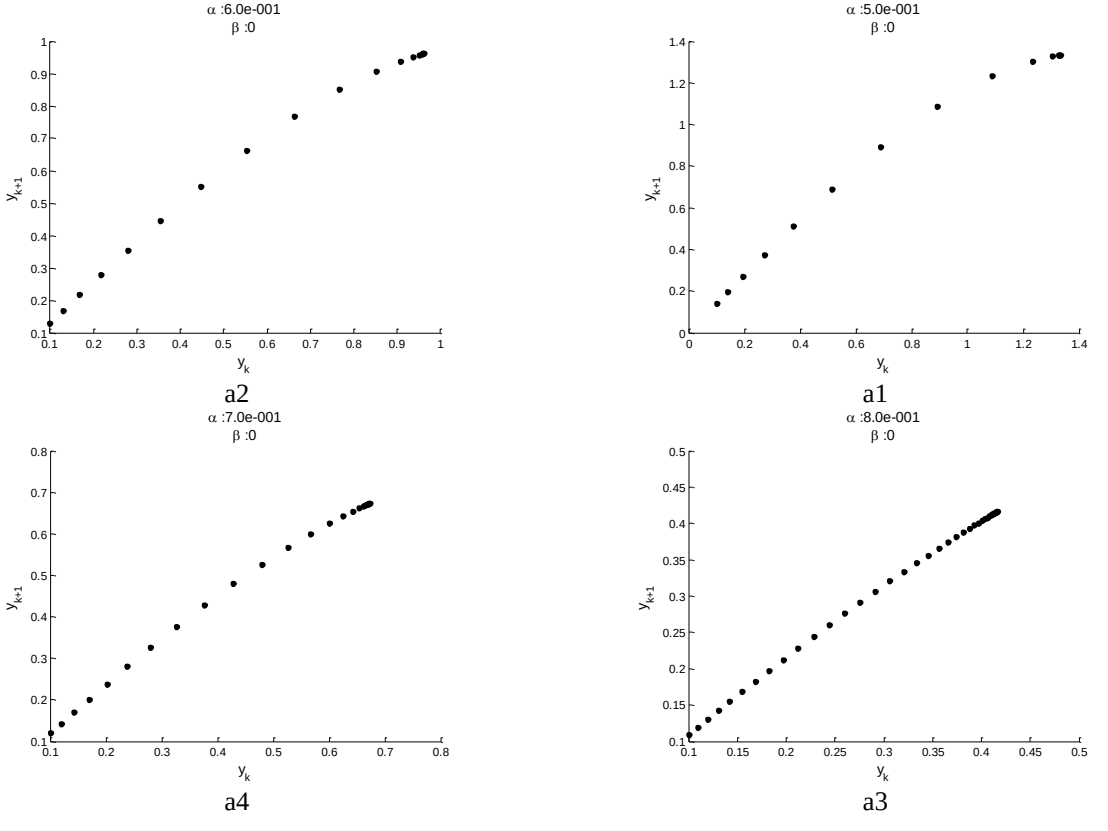


Fig.10 Phase Plane of Poincare Map for $x = \alpha y$ with different α (negetive value)



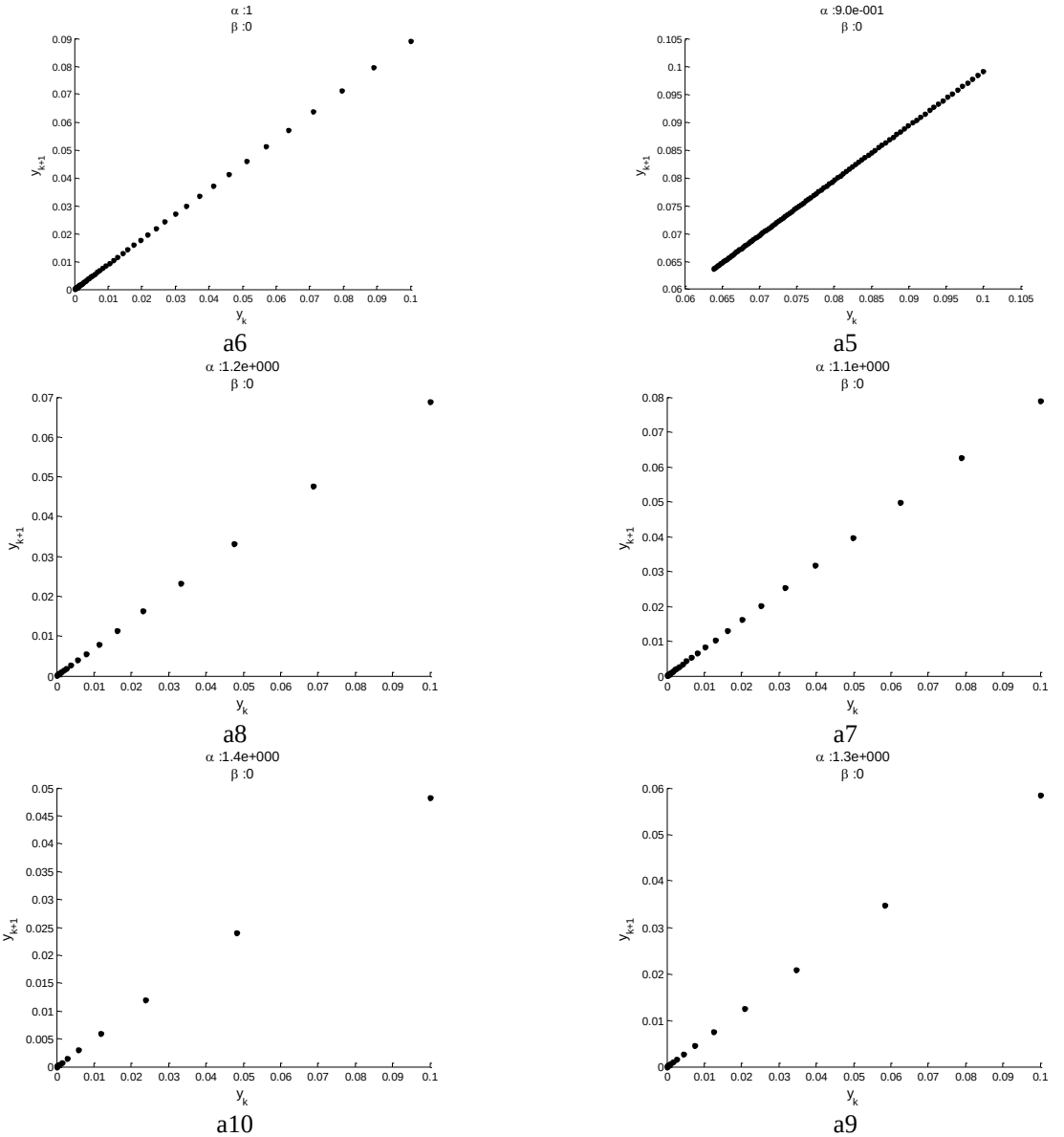


Fig.11 Phase Plane of Poincare Map for $x = \alpha y$ with different α (Posetive value)

REFERENCES

[1] D.F. Griffith, D.J. Higham, ***Numerical Methods for Ordinary Differential Equations***, Springer 2010

[2] S. P. Hastings and J. B. McLeod ***Classical Methods in Ordinary Differential Equations: With Applications to Boundary Value Problems***, Amer. Math. Soc., Rhode Island, 2011

[3] K. Atkinson, W. Han, ***Elementary Numerical Analysis***, John Wiley, Sons, Inc. 2004.

[4] J. Stoer, R. Bulirsch, ***Introduction to numerical Analysis***, Springer-Verlag, 2002.

[5] Paul Blanchard, Robert L. Devaney, Glenn R. Hall, ***Differential Equations*** (Preliminary Edition), PWS Publishing, Boston, 1996.

[6] V.I. Arnold, ***Ordinary Differential Equations***, Springer, Berlin, 1992.

[7] E.A. Coddington and N. Levinson, ***Theory of Ordinary Differential Equations***, McGraw-Hill, New York, 1955.

[8] P. Hartman, ***Ordinary Differential Equations***, 2nd ed., SIAM, Philadelphia, 2002.

[9] L. Perko, ***Differential Equations and Dynamical Systems***, 2nd ed., Springer, New York, 1996

[10] F. Verhulst, Nonlinear, ***Differential Equations and Dynamical Systems***, 2nd ed., Springer, Berlin, 2000

[11] Piskunov, Nikolaĭ Semenovich. ***Differential and integral calculus***. P. Noordhoff, 1965.

[12] Nagle, Saff& Snider ***Fundamentals of Differential Equations***, 6th Ed, by, Addison-Wesley, 2004

[13] Alexey N. Pavlov, Dmitry V. Dumsky 'Return times dynamics: **role of Pointcare Section in numerical Analysis' Chaos, Solitons and Fractals** 18.3 (2003):795-801

[14] Awrejcewicz, Jan, and Peter Hagedorn, eds. ***Nonlinearity, Bifurcation and Chaos: Theory and Applications*.** InTech, 2012.

[15] Hegger, Rainer, Holger Kantz, and Thomas Schreiber. "**Practical implementation of nonlinear time series methods: The TISEAN package.**" *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 9.2 (1999): 413-435.

[16] Hilborn, R.C., ***Chaos and Nonlinear Dynamics: An Introduction for Scientists and Engineers***. Second Edition ed, 2000.

[17] Schuster, H.G., ***Deterministic Chaos: an Introduction***. Weinheim: VCH, 1989

[18] Pikovsky, A.S., M.G. Rosenblum, and J. Kurths, Synchronization. ***A Universal Concept in Nonlinear Sciences***., Cambridge: Cambridge University Press., 2001

[19] Ott, E., ***Chaos in Dynamical Systems***., Cambridge: Cambridge University Press, 1994.