Robust Optimal Control of Two-Wheeled Self-Balancing Robot using Chebyshev Inclusion Method

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Received: October 2017 Revised: December 2017 Accepted: January 2018

ABSTRACT:
Since two-wheeled and the self-balancing robot has a complicated and non-linear structure, its model has some uncertainties. These uncertainties cause that the system has an incorrect solution while using the classic methods for controlling of it. In this paper, a new method based on interval analysis is proposed for modeling the optimal control of the two-wheeled and self-balancing robot with interval uncertain parameters which require only lower and upper bounds of uncertain parameters, with no need to know about probability distributions. Since the system has uncertainties in it, controllability is first analyzed based on interval arithmetic. Afterwards, LQR based method based on Pontryagin principle is utilized to solve the problem. Finally, by solving the interval Ricatti equations, the confidence interval for feedback controller has been achieved. Final results are compared with Monte Carlo method and the results demonstrate the effectiveness of the proposed method.

KEYWORDS: Optimal control, Interval analysis, LQR, Chebyshev inclusion method, Monte Carlo, Two-Wheeled Self-Balancing Robot.

1. INTRODUCTION

In most applications, optimal control problems (OCPs) are considered as deterministic problems [1]. Although, this assumption is not valid practically and there are always some uncertainties in the system parameters which have to be taken into account during the optimization process. These uncertainties can be made for different reasons like measurement mistakes, having insufficient information about the system nature, neglecting some unknown parameters, unrecognized dynamics, etc. [2].

These uncertainties make it impossible to solve the OCPs by deterministic methods; i.e. the final solution may give the wrong solution due to uncertainties on the system dynamics with no well convincing solution for the desired control rule. Hence, conventional deterministic methods may be failed in solving the problems with uncertainties.

This problem leads researchers to study different methods under uncertainties [3]. Recently, some different methods are introduced to overcome this objection. Stochastic, fuzzy programming and interval arithmetic methods are among the most popular methods for problems under uncertainties [4-7].

For example, fuzzy methods are based on expert experiences (membership functions); i.e. this method will fail if there is no (less) information about the system dynamics. In stochastic methods, probabilistic distribution is required to utilize the system dynamics. Hence, when there is no information about membership functions and probabilistic distributions, the best solution is to utilize the interval method.

The main purpose of this study is to use the interval analysis in the optimal control problems. In the interval based methods, only having information about the upper and the lower bounds is needed. Interval methods have been introduced over the years [8] but from 1996, the interval arithmetic was introduced in simple propositions [9].

In this paper, an interval extension of the LQR optimal control is proposed and its application is illustrated on a two-wheeled robot system with uncertainties. In 2013, Wu et al. introduced a method for solving ODE systems by a new orthogonal interval system, Chebyshev inclusion method. In this study, we propose this method for optimal control problems and utilize it for a practical case study.

The present considered practical case study is a two-wheeled and self-balancing robot. This robot belongs to a nonlinear, complicated and unstable essential motion control system which is considered as one of the typical case studies in control. Since the considered robot has more advantages in different aspects of stable running, simple structure, strong environmental adaption, and high energy utilization rate, it has lots of applications from civilian fields to military fields.

Since the 1980s, there are different studies about two-wheeled self-balancing robots. In 2017, Qiu et al. introduced a fuzzy control based system for the two-wheeled self-balancing robot system; this technique overcomes the instability and nonlinear nature of the system, but it relies on the expert's experience too much [10].

However, they do not consider any uncertainties which can be made based on incorrect system identification. This objection is studied in this paper.

Consider a mathematical model of two-wheeled self-balancing robots with interval uncertainties. The main purpose is to design an optimal feedback controller for the system so that it stays stable in the designed confidential interval.

There are some researchers which have worked on the feedback control of systems with uncertain intervals [11-14]. But, the final results for these methods were a real-valued controller with no confidence interval. This problem leads us to design a new feedback controller based on interval analysis.

Different methods of robust control are created after the appearance of Kharitonov Theorem which utilized the interval transfer function to model the system dynamics [15-17]. The application of their methods are limited; because transfer function representation can be made it difficult or even impossible to solve some classes of the dynamical objects especially those who need to be approximated by state space models.

The novelty of this study is that the interval optimal control based on Riccati equation as a state representation of the dynamical systems is proposed. An interval-based controller is also designed to achieve the confidential interval and finally, this new method is applied to a practise based mathematical model of a two-wheeled self-balancing robot.

Since the system has uncertainties, the system is first analyzed from the point of controllability and then the proposed method is applied to it.

This method does not guarantee to be globally optimal. However, the results provide guaranteed lower and upper bounds of the costs which are necessary to perform the control task.

2. INTERVAL ANALYSIS

This section includes some introduction to the interval analysis. System uncertainties can be defined by the interval integers. An interval uncertainty can be bounded by a set of real integers such that they include all the possible uncertainties of the system. Consider a real interval integer $\tilde{X}$; this interval integer can be defined as follows:

$$\begin{aligned}\tilde{X} &= [\underline{x}, \overline{x}] \\ &= \{x \mid x \in I(\mathbb{R}) \cup \{-\infty, \infty\}, \underline{x} \leq x \leq \overline{x}\}, \\ I(\mathbb{R}) &= \{\tilde{X} \mid \tilde{X} = [\underline{x}, \overline{x}],\ \underline{x}, \overline{x} \in \mathbb{R},\ \underline{x} \leq \overline{x}\},\end{aligned} \tag{1}$$

where $\tilde{X}$ is an interval integer over $I(\mathbb{R})$ and $\underline{x}$ $\overline{x}$ are the lower and upper bounds of $\tilde{X}$ respectively. The primary interval operations between two interval integers $\tilde{A}$ and $\tilde{B}$ can be described as follows:

$$\tilde{A} + \tilde{B} = [(\tilde{A}_c + \tilde{B}_c) - \mathrm{k},\ (\tilde{A}_c + \tilde{B}_c) + \mathrm{k}] \tag{2}$$

$$\tilde{A} - \tilde{B} = [(\tilde{A}_c - \tilde{B}_c) - \mathrm{k},\ (\tilde{A}_c + \tilde{B}_c) + \mathrm{k}] \tag{3}$$

$$\tilde{A} \times \tilde{B} = [\tilde{A}_c \times \tilde{B}_c - \eta,\ \tilde{A}_c \times \tilde{B}_c + \eta] \tag{4}$$

$$\begin{aligned}&\tilde{A} / \tilde{B} = \tilde{A} \times \frac{1}{\tilde{B}}, \\ &\frac{1}{\tilde{B}} = [\frac{1}{\tilde{B}_c} - \gamma, \frac{1}{\tilde{B}_c} + \gamma],\ 0 \notin [\underline{b}, \overline{b}]\end{aligned} \tag{5}$$

where,

$$\tilde{A}_c = a_{i,j} = \frac{1}{2}(\underline{a}_{i,j} + \overline{a}_{i,j}) \tag{6}$$

$$k = \frac{(\overline{x} + \overline{y}) - (\underline{x} + \underline{y})}{2}, \tag{7}$$

$$\eta = [\min\{\tilde{A}_c \times \tilde{B}_c\} - \alpha,\ \beta - \max\{\tilde{A}_c \times \tilde{B}_c\}] \tag{8}$$

$$\alpha = \min\{\underline{a}\,\underline{b}, \overline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\overline{b}\}, \tag{9}$$

$$\beta = \max\{\underline{a}\,\underline{b}, \overline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\overline{b}\}, \tag{10}$$

$$\gamma = \{\frac{1}{\overline{b}}(\frac{\overline{b} - \underline{b}}{\overline{b} + \underline{b}}),\ \frac{1}{\underline{b}}(\frac{\overline{b} - \underline{b}}{\overline{b} + \underline{b}})\}, \tag{11}$$

More detailed on the interval arithmetic can be found in [9, 18]. An interval can be also extended to an interval matrix (vector). An interval matrix $\tilde{A}$ can be defined as follows:

$$\begin{aligned}&\tilde{A} = \{a_{i,j} \in R^{m \times n} : \underline{a}_{i,j} \leq a_{i,j} \leq \overline{a}_{i,j}\}, \\ &i, j = 1, 2, \ldots, n.\end{aligned} \tag{12}$$

Matrix operations for interval integers are like the real integers (non-interval); that is:

$$\tilde{A} \pm \tilde{B} = a_{i,j} \pm b_{i,j} \tag{13}$$

$$\tilde{A} \times \tilde{B} = \sum_{k=1}^{P} a_{i,k}\, b_{k,j} \tag{14}$$

$i, j = 1, 2, \ldots, n$ (15)

Monotonic subset property for addition, subtraction, and multiplication in between interval matrices holds.

From the classical integer mathematics, we know that a matrix will be invertible if and only if its determinant does not contain zero. Similarly, in the interval analysis, the square interval matrix $\tilde{A}$ will be invertible if its determinant does not contain interval zero ($\{0\} = [0,0] \notin |\tilde{A}|$).

Proposition 1 [19]: if $\tilde{A}$ is invertible, the matrix equation $\tilde{A}\tilde{X} = \tilde{I}$ and $\tilde{X}\tilde{A} = \tilde{I}$ both will have a single solution: $\tilde{X} = \frac{1}{|\tilde{A}|}\tilde{A}$.

Proposition 2 [20]: let $\tilde{A} = [\underline{a}, \overline{a}] \in \mathrm{IR}^{n\times n}$. if $\underline{a}$ and $\overline{a}$ are regular and $\underline{a}^{-1} \geq 0$ $\overline{a}^{-1} \geq 0$, then $\tilde{A}$ is regular and $\tilde{A}^{-1} = [\overline{a}^{-1}, \underline{a}^{-1}] \geq 0$.

3. CHEBYSHEV INCLUSION METHOD

Wu et al. introduced an inclusion based Chebyshev function with narrower estimation of the bounds toward the interval function with overestimation in interval arithmetic [21]. A brief introduction about this method is introduced in the following.

3.1. Chebyshev Approximation Method

Let's consider a continuous function $f(x)$ in the closed interval $[a,b]$; for any $\varepsilon > 0$, there is a polynomial, $p(t)$ which can uniformly approximate the function $f(x)$ as closely as desired by a polynomial function such that,

$$\| f(x) - p(x) \|_{\infty} < \varepsilon,\ x \in [a,b]. \tag{16}$$

The above definition is known as the Weierstrass theorem [22, 23]. Making a balance between computational cost and accuracy in polynomials is significant [24].

Consider the Chebyshev polynomial (Ci) which is defined in the interval $t \in [-1,1]$ of degree I [25]:

$$C_i(t) = \cos i\theta, \theta = \arccos(t) \in [0,\pi] \tag{17}$$

where i is a nonnegative integer.

The utilized Chebyshev series for approximation, *f*(*x*) is a truncated version with an accuracy of degree *k*:

$$\begin{aligned} f(t) \approx p(t) &= \frac{1}{2} f_0 + \sum_{i=1}^{k} f_i C_i(t) \\ &= \frac{1}{2} f_0 + \sum_{i=1}^{k} f_i \cos(i\theta) \end{aligned} \tag{18}$$

where f_i is the i^{th} constant coefficient. By applying the Mehler (Gaussian-Chebyshev) integration formula [26] on the formula, the coefficients of Chebyshev polynomial are achieved as follows:

$$f_i = \frac{2}{p}\sum_{j=1}^{p} f(\cos\theta_j)\cos i\theta_j \tag{19}$$

where *p* shows the truncation order.

3.2. Chebyshev Inclusion Method

Using interval analysis for generating the truncated Chebyshev polynomial results a high accuracy method which can control the overestimation efficiently [27]. This process turns the scalar t into the interval integer [t]. From [27], using the trigonometric representation instead of the polynomial representation for the Chebyshev polynomials in the interval arithmetic results better controlling over the overestimation. By considering the trigonometric representation, the Chebyshev inclusion function can be described by the following formula:

$$[f_{C_k}] = \frac{1}{2} f_0 + \sum_{i=1}^{k} f_i \cos(i[t]) \tag{20}$$

where, $[t] \in [-1,1]$ and $[\theta] = \arccos([t]) = [0,\pi]$. Since $[\cos](i[\theta]) = [\cos]([0,i\pi]) = [-1,1]$.

The Chebyshev inclusion polynomial can be reformulated as follows:

$$[f_{C_k}] = \frac{1}{2} f_0 + [-1,1] \times \sum_{i=1}^{k} | f_i | \tag{21}$$

Although the Chebyshev inclusion method is not the rigorous solution and it neglects the truncated error and numerical error of integration error, these errors can be neglected compared with the overestimation of the interval analysis.

4. MONTE CARLO METHOD

Monte Carlo (MC) method is a class of computational algorithms which depends on repeated random sampling to achieve the numerical results. The essential idea in MC is to use the randomness for solving the problems that might be deterministic in principle. Because of its easy implementation, it is usually employed in most of physical and engineering applications. In MC method, random variable samples are taken based on the probability distribution and then the probability distribution of response is computed directly as the output. The accuracy of Monte Carlo has a direct relationship with the sampling size; in other words, by considering the weak law of large number and N as the sampling size, the convergence ratio is $N^{-\frac{1}{2}}$ [28].

5. APPROXIMATED TWO-WHEELED AND SELF-BALANCING ROBOT WITH INTERVAL UNCERTAINTIES

The two-wheeled and self-balancing robot which is studied in this paper is a chassis mounted on the top of an axel incorporating two wheels where the chassis has no balancing support. Since the regarded robot can be considered as the vehicle-mounted inverted pendulum, the dynamic system analysis process is more complex.

One of the important problems which are usually not considered in the mathematical modeling of two-wheeled self-balancing robot is neglecting its some unknown terms, resistant value, etc.

This problem causes the researchers to have a deviation in the design. In this paper, a new method based on interval analysis is utilized to consider the robot identification uncertainties. This method will be made the controller more robust and more practical against the condition changes.

In this paper, we first separate the wheel from the pendulum analysis in the process of modeling [29], and then, the possible uncertainties are added to the system model.

Fig. (1) shows a diagram revolver force analysis. According to the revolver, force equation can be obtained in the following according to Newton's law [30] and the rotational torque formula [31]:

$$M_w \ddot{x} = H_{fR} - H_R, \qquad (22)$$
$$I_w \ddot{\theta}_w = C_R - H_{fR}.R.$$

where M_w represents the weight of the wheel, I_w is the moment of inertia of the wheel; *R* represents the radius of the wheel, $\ddot{x}$ is the wheel acceleration of the *x*-axis.-*Z*is the H_R rque and is the right wheel to C_R axis force of the right wheel with the car body. H_{fR} is the inter-atomic force of the right wheel with the ground and θ_w is the angle of the wheel around the *Z*-axis direction.

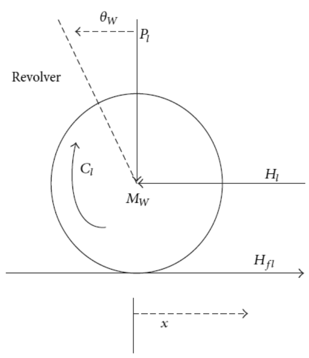


Fig. 1. The force analysis of revolver [29].

Similarly, for the right wheel, the force equation can be achieved by the following formula:

$$M_w \ddot{x} = H_{LR} - H_R, \qquad (23)$$
$$I_w \ddot{\theta}_w = C_L - H_{fL}.R.$$

where C_L is the left wheel torque and H_L is the *Z*-axis force of the left wheel with the car body. H_{fL} is the inter-atomic force of the left wheel with the ground. Since the final Equation will be achieved as follow:

$$2(M_w + \frac{I_w}{R^2})\ddot{x} = \frac{C_R + C_L}{R} - (H_R + H_L). \qquad (24)$$

After using Newton's second law for the horizontal and vertical forces [29], the final approximated interval system is achieved as follows:

$$(M_p + 2M_W \frac{I_W}{R^2})\ddot{x} = \frac{C_R + C_L}{R} - 2M_P l\theta_P, \qquad (25)$$
$$(2M_p l^2 + I_P)\ddot{\theta}_P = M_P g l \theta_P - M_P l \ddot{x}.$$

where θ_P represents the angle of the car body deviating from the *Z*-axis direction, M_P is the weight of the car body, I_P is the moment of inertia of the car body and l is the height of which the car body is apart from the shaft.

In this equation, the output torque for the wheel is $C_R = C_L = I_R(dw/dt) = (k_m/R)U_a - (-k_m k_e/R)\dot{\theta}_w$; finally, the state space representation of the two-wheeled self-balancing robot is achieved as follows:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_p \\ \ddot{\theta}_p \end{bmatrix} = \begin{bmatrix} \{0\} & \{1\} & \{0\} & \{0\} \\ \{0\} & \frac{2k_m k_e(M_p lr - I_P - M_P l^2)}{Rr^2 A} & \frac{M_P{}^2 g l^2}{A} & \{0\} \\ \{0\} & \{0\} & \{0\} & \{1\} \\ \{0\} & \frac{2k_m k_e(rB - M_P l)}{Rr^2 A} & \frac{M_P{}^2 g l B}{A} & \{0\} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta_p \\ \dot{\theta}_p \end{bmatrix} + \begin{bmatrix} \{0\} \\ \frac{2k_m(I_P + M_P l^2 - M_P lr)}{RrA} \\ \{0\} \\ \frac{2k_m(M_P l - rB)}{RrA} \end{bmatrix} U_a. \qquad (26)$$

And the output state representation is:

$$y = [\{0\}\ \{0\}\ \{1\}\ \{0\}] \begin{bmatrix} x \\ \dot{x} \\ \theta_p \\ \dot{\theta}_p \end{bmatrix}. \qquad (27)$$

where, $A = [I_P \beta + 2M_P l^2(M_w + (I_w/r^2))]$ and $B = [2M_w + (2I_w/r^2) + M_P$, and the terms $\{0\} = [0,0]$ and $\{1\} = [1,1]$ are called degenerate integers [32].

The final interval state representation of the system with uncertainties is given in the following formula.

$$\tilde{A}=\begin{bmatrix}\{0\} & \{1\} & \{0\} & \{0\}\\ \{0\} & [-0.25,-0.11] & [24.6,56.05] & \{0\}\\ \{0\} & \{0\} & \{0\} & \{1\}\\ \{0\} & [-0.61,-0.49] & [237,239] & \{0\}\end{bmatrix},\quad (28)$$

$$\tilde{B}=\begin{bmatrix}\{0\}\\ [0.41,0.56]\\ \{0\}\\ [1.9,2.72]\end{bmatrix},\ \tilde{C}=\begin{bmatrix}\{0\} & \{0\} & \{1\} & \{0\}\end{bmatrix}.$$

6. PROBLEM STATEMENT (INTERVAL QUADRATIC REGULATOR)

Consider a linear multivariable state-space model of the plant dynamics with interval uncertainties as follows:

$$\dot{x}(t)=\tilde{A}\,x(t)+\tilde{B}\,u(\mathrm{t}),\quad (29)$$

where $x(t)\in\mathbb{R}^n$ is state vector and $u(\mathrm{t})\in\mathbb{R}^p$ is an input vector. The elements $a_{i,j},\ b_{i,j}\ \ (i,j=1,2,\ldots,n;k=1,2,\ldots,p)$ of matrix $\tilde{A}\in I(\mathbb{R}^{n\times n})$ and matrix $\tilde{B}\in I(\mathbb{R}^{n\times p})$ are interval integers bounded by a defined upper and lower interval; i.e. $\tilde{A}=[\underline{A},\overline{A}]$ and $\tilde{B}=[\underline{B},\overline{B}]$ are interval system matrix and input matrix where their elements lie between upper and lower bounds.

The boundary conditions of the system are:

$$x(t_0)=X_0,\ x(t_f)=X_f.\quad (30)$$

where, X_0 and X_f describes the initial and the final states of the system respectively. Consider the performance measure as follows:

$$J(x(t),u(\mathrm{t}),\Delta)=\frac{1}{2}x^T(t_f)\,\mathrm{F}(t_f)\,x(t_f)+\frac{1}{2}\int_{t_0}^{t_f}[x^T(t)\ u^T(t)]\begin{bmatrix}\tilde{Q}(t) & 0\\ 0 & \tilde{R}(t)\end{bmatrix}\begin{bmatrix}x(t)\\ u(t)\end{bmatrix}dt,\quad (31)$$

In this case, Δ shows that the system has interval uncertainties, $\tilde{Q}(t)$ is positive semi-definite and $\tilde{R}(t)$ is positive definite interval matrices and $J(t,x(t),u(\mathrm{t}),\Delta)=[\underline{j}(t,x(t),u(\mathrm{t})),\ \overline{j}(t,x(t),u(\mathrm{t})]$ describes the interval-valued performance index.

By expanding the interval arithmetic into the Pontryagin principle, the interval Hamiltonian equation of the problem is achieved as follows:

$$\tilde{H}(x(t),u(t),\lambda(t))=\frac{1}{2}x^T(t)\tilde{Q}x(t)+\frac{1}{2}u^T(t)\tilde{R}u(t)+\lambda(t)(\tilde{A}\,x(t)+\tilde{B}\,u(\mathrm{t})),\quad (32)$$

By applying the optimal control on the interval Hamiltonian matrix,

$$\frac{\partial\tilde{H}}{\partial u}=\tilde{0}\rightarrow\tilde{R}u(t)+\tilde{B}^T\lambda=0,\quad \Rightarrow\tilde{u}^*(t)=-\tilde{R}^{-1}\,\tilde{B}^T\lambda,\quad (33)$$

$$\dot{x}(t)=\frac{\partial\tilde{H}}{\partial\lambda}\rightarrow\dot{x}(t)=\tilde{A}\,x(t)+\tilde{B}\,u(\mathrm{t})\quad (34)$$

$$\dot{\lambda}(t)=-\frac{\partial\tilde{H}}{\partial x}\rightarrow\dot{\lambda}(t)=-\tilde{Q}\,x(t)-\tilde{A}^T\,\lambda(t)$$

That is:

$$\begin{bmatrix}\dot{x}(t)\\ \dot{\lambda}(t)\end{bmatrix}=\begin{bmatrix}\tilde{A} & -\tilde{E}\\ -\tilde{Q} & -\tilde{A}^T\end{bmatrix}\begin{bmatrix}x(t)\\ \lambda(t)\end{bmatrix},\quad (35)$$

$$\tilde{E}=\tilde{B}\,\tilde{R}^{-1}\tilde{B}^T,$$

Note that if $\tilde{R}$ is a matrix (i.e. multi-input systems), the inverse matrix of the equation should be first achieved by the explained interval inverse matrix method.

For closed-loop optimal control, we assume $\lambda(t)=\tilde{P}(t)\,x(t)$.Since,

$$\tilde{u}^*(t)=-\tilde{R}^{-1}\tilde{B}^T\tilde{P}\,x(t)=-\tilde{k}\,x(t)\quad (36)$$

and,

$$\begin{cases}\dot{x}(t)=\tilde{A}\,x(t)-\tilde{B}\,\tilde{R}^{-1}\tilde{B}^T\tilde{P}x(t)\\ \dot{\lambda}(t)=-\tilde{Q}\,x-\tilde{A}^T\tilde{P}\,x(t)\end{cases}\quad (37)$$

By solving the equation above, the final equation will become as follows:

$$\tilde{P}\tilde{A}+\tilde{A}^T\tilde{P}+\tilde{P}\tilde{B}\,\tilde{R}^{-1}\tilde{B}^T\tilde{P}x+Q+\dot{\tilde{P}}=0\quad (38)$$

This equation is the interval matrix extension of the Algebraic Riccati Equation (*IMARE*), where the interval solution $\tilde{P}$ is required to achieve the optimal interval feedback gain $\tilde{k}$ such that $\tilde{k}=\tilde{R}^{-1}\tilde{B}^T\tilde{P}$.

By simplification of the achieved IMARE, some interval ordinary differential equations (IODEs) have been extracted.

Since an interval method is required to solve the IODEs. Generally, the interval methods experience overestimation in the computation because of its intrinsic wrapping effect [33]. This shortcoming leads researchers to work on methods for reducing the overestimation of the interval calculations [34, 35].

In this study, the Chebyshev inclusion function is utilized because of its ability in tightening the interval bounds [3].

More details about Chebyshev inclusion function is given in the previous section.

From the above, it is clear that the final solutions result in interval values for feedback coefficients; that is, a confident bound which can provide the considered sub-optimal controller under interval uncertainties. After achieving the feedback gain coefficients, the final feedback law can be achieved from the Eq. 36.

7. APPLYING THE PROPOSED METHOD ON TWO-WHEELED AND SELF-BALANCING ROBOT WITH INTERVAL UNCERTAINTIES

Consider an optimal control problem for the two-wheeled and self-balancing robot motor speed model with interval uncertainties which is described in the previous section; a linear differential equation with n = 4, p = 1 and interval matrices $\tilde{A}$ and $\tilde{B}$. Despite the quadratic systems which have controllability guarantees, because of the system uncertainties, we need to check its constructability based on interval methods like [36].

7.1. Controllability Test for the Proposed Two-Wheeled and Self-balancing Robot with Interval Uncertainties

Consider the Eq. (31). This system will be controllable if, for the initial condition of $x(0) = x_0$ and for any given vector x_f, there exist a limited time like t_f and input $u(\mathrm{t})$ in the interval $[0, t_f]$ where this input maps the system from the x_0 into x_f in time t_f; i.e. $x(t_f) = x_f$; otherwise, the given equation is uncontrollable. There are different methods for analyzing the controllability of the systems with real integer values [37]. In this section, we utilized an interval based method from Shashikhin [36]. The method can be summarized as follows:

1) Compute controllability matrix $\tilde{D}$ for the interval pair $(\tilde{A}, \tilde{B})$:

$$\tilde{D} = [\tilde{B} \mid \tilde{A}\tilde{B}, \ldots, \tilde{A}^{n-1}\tilde{B}]$$

2) Generate Matrix $\tilde{D}\tilde{D}^T$.
3) Determine the median of $\tilde{D}\tilde{D}^T$.
4) Determine the singular values and vectors of the interval matrix $\tilde{D}\tilde{D}^T$ ($\lambda_i(med(\tilde{D}\tilde{D}^T)),\ i = 1, \ldots, n-1$).
5) Find the null space of the interval matrix $null(med(\tilde{D}\tilde{D}^T)) = \varepsilon_i, i = 1, \ldots, n-1$
6) Find interval singular values:

$$\tilde{\lambda}_i(\tilde{D}\tilde{D}^T) = [\lambda_i(med(\tilde{D}\tilde{D}^T)) - \varepsilon_i, \lambda_i(med(\tilde{D}\tilde{D}^T)) + \varepsilon_i], i = 1, \ldots, n-1$$

7) If all the interval singular values belong to positive and definite interval set, the order of the interval set is perfect and the system is controllable.

From the above, for the case study we have:

$$\tilde{D} = [\tilde{B} \mid \tilde{A}\tilde{B} \mid \tilde{A}^2\tilde{B} \mid \tilde{A}^3\tilde{B}] = \begin{bmatrix} \{0\} & [0.41, 0.56] & [-0.14, -0.045] & [46.74, 152.49] \\ [0.41, 0.56] & [-0.14, -0.0451] & [46.74, 152.46] & [-57.27, -10.09] \\ \{0\} & [1.9, 2.72] & [-0.34, -0.20] & [450.32, 650.16] \\ [1.9, 2.72] & [-0.34, -0.2] & [450.32, 650.16] & [-174.66, -70.51] \end{bmatrix}. \quad (39)$$

$$\tilde{D}\tilde{D}^T = \begin{bmatrix} [2185, 23254] & [-8754.5, -4737] & [21049, 91440] & [-26725, -3316] \\ [-8754.5, -473.7] & [2287, 26529] & [-37287, -4553] & [21760, 109150] \\ [21049, 99144] & [-37287, -4553] & [202790, 422720] & [-113780, -31840] \\ [-26640, -3316] & [21760, 108830] & [-113780, -31840] & [207760, 453220] \end{bmatrix}. \quad (40)$$

By achieving the median value,

$$med(\tilde{D}\tilde{D}^T) = \begin{bmatrix} 12720 & -6746 & 56245 & -15021 \\ -4614 & 14408 & -20920 & 65455 \\ 60097 & -20920 & 312755 & -72810 \\ -14978 & 65295 & -72810 & 330490 \end{bmatrix}. \quad (41)$$

After applying the formula in the flowchart, the Eigenvalues and the null values are achieved as follows:

$\lambda_1 = 411360;\ \varepsilon_1 = 3.6007e^{-10}$

$\lambda_2 = 255930;\ \varepsilon_2 = 2.8086e^{-10}$

$\lambda_3 = 3170;\ \varepsilon_3 = 1.6130e^{-11}$

$\lambda_4 = 470;\ \varepsilon_4 = 4.7749e^{-11}$

And the final interval Eigenvalues are $\tilde{\lambda}_1 = [411360, 411360]$, $\tilde{\lambda}_2 = [3170, 3170]$, $\tilde{\lambda}_3 = [255930, 255930]$ and $\tilde{\lambda}_4 = [470, 470]$.

As it can be seen, the entire interval Eigenvalues are positive; so, the system is controllable.

Note that because of the Low amount of epsilon and the high amount of the Eigenvalues, the lower and the upper bounds of the interval Eigenvalues are similar to the degenerate integers.

7.2. Optimal Interval Control of the Two-Wheeled and Self-Balancing Robot

To design the optimal control of the considered interval LQR, we should first specify the augmented system matrices [38]. From the previous subsection, we have $\tilde{A}$ and $\tilde{B}$ matrices. The other required matrices are given in the following:

$$\tilde{Q}(t)=\begin{bmatrix} [220,223] & \{0\} & \{0\} & \{0\} \\ \{0\} & [168,172] & \{0\} & \{0\} \\ \{0\} & \{0\} & [120,122.2] & \{1\} \\ \{0\} & \{0\} & \{0\} & [187,188.6] \end{bmatrix}, \quad (42)$$

$\tilde{R}=\{1.77\}$.

By considering the given matrices and generating the Interval Algebraic Riccati Equation and solving these equations using the proposed interval Chebyshev method, the optimum feedback gain ($K=[\underline{k},\overline{k}]$) is obtained as follows:

$\underline{k}=[-11.19,-16.86,221.39,19.90]$

$\overline{k}=[-11.19,-15.20,295.69,23.55]$.

Here, we consider all the confidence intervals for preventing the system from the sudden problems. In the following, the simulation results of the proposed method and it's comparison with the Monte Carlo is given. The iteration for the Monte Carlo is 100.

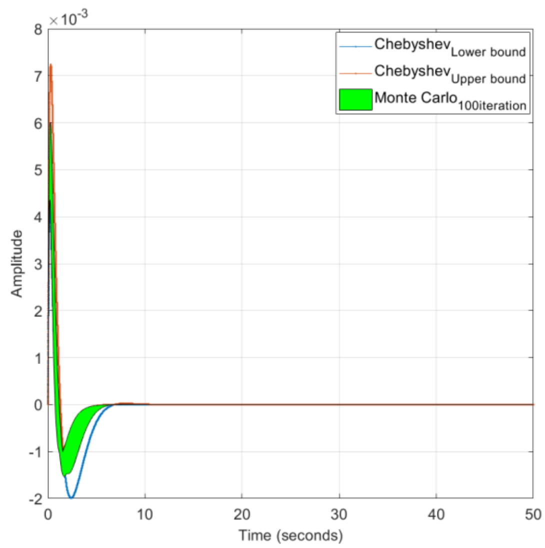


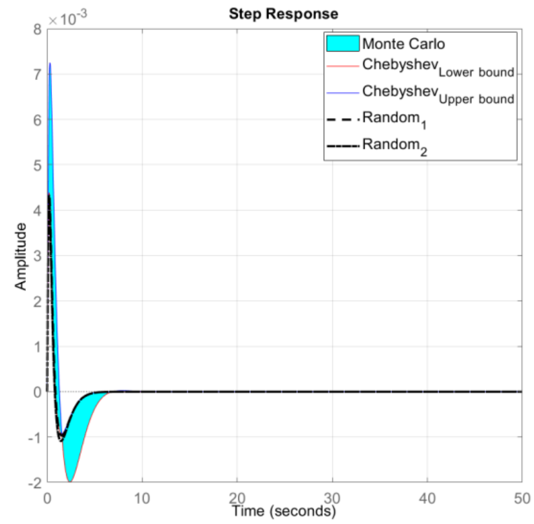
Fig. 2. Comparison of step response for proposed Interval Chebyshev method and MC method (100 iterations) for the two-wheeled and self-balancing robot with interval uncertainties.

From the above results, it can be seen that while the upper bound of the proposed method and MC with 100 iterations is equal, the lower bound of the proposed method includes more surface than the MC. The reason is that the MC method depends on the number of its iterations.

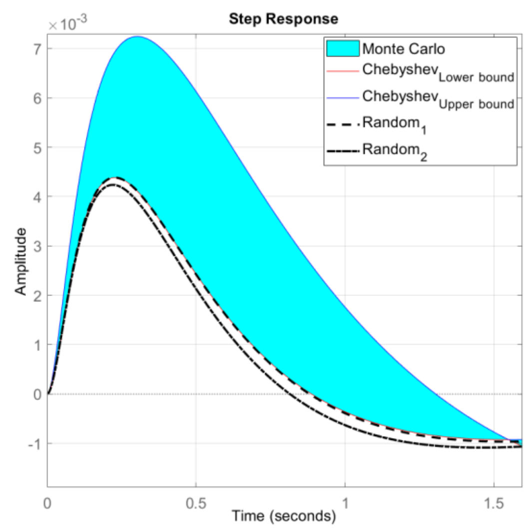
In the following, we consider two random values for the system close to the lower bound and analyze the results. From the figure, it can be seen that the *Random₁* is included in the lower bound of both MC and the proposed method and the *Random₁* has a small error (about $1e-3$) in the lower bound for about 1.5 sec. and then it stands on the guaranteed interval.

Table 1. Running time of the proposed method.

Method	Chebyshev inclusion method	MC method
Running time (sec.)	1.2	30



(a)



(b)

Fig. 3. Step response for two-wheeled and self-balancing robot with uncertainties by MC and Chebyshev inclusion methods and two random inputs: (a) in the time interval [0,50] and (b) [0,1.5].

8. CONCLUSION

An interval extension of the linear quadratic regulator is introduced for optimal control of two-wheeled self-balancing robot. The proposed method is developed based on Chebyshev polynomials to solve the final interval Riccati equations. The interval-based method is utilized for optimal control problem with uncertain-but-bounded parameters, without requiring complete information. Indeed, this method is a robust method for systems with uncertainties. The proposed Chebyshev inclusion method compared with the Monte Carlo method and the ability of the proposed method has been shown in the results. The simulation experiments indicate that the proposed method can realize the self-balance robust control of the two-wheeled robot successfully.

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